Feynman Lectures on the Strong Interactions

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These twenty-two lectures, with exercises, comprise the extent of what was meant to be a full-year graduate-level course on the strong interactions and QCD, given at Caltech in 1987-88. The course was cut short by the illness that led to Feynman's death. Several of the lectures were finalized in collaboration with Feynman for an anticipated monograph based on the course. The others, while retaining Feynman's idiosyncrasies, are revised similarly to those he was able to check. His distinctive approach and manner of presentation are manifest throughout. Near the end he suggests a novel, nonperturbative formulation of quantum field theory in D dimensions. Supplementary material is provided in appendices and ancillary files, including verbatim transcriptions of three lectures and the corresponding audiotaped recordings.

Contents

Preface

During the last year of my Ph.D. at Caltech in 1987-88, I was looking for a course to TA that would not take too much time from finishing my dissertation. I had heard that Feynman did not assign homework in his courses, and in my naiveté asked him if I could be his teaching assistant for a new course that had been announced, on quantum chromodynamics. After checking my credentials with my supervisor John Preskill, he agreed. Only afterwards did I realize that the TA in Feynman's courses was generally the person who did the transcription of the notes to create the monograph that would follow. This was not the easy job I had bargained for, and I persuaded Steven Frautschi to assign several other TAs to the course to divide the labor. We took turns rewriting the lectures into publishable form, which Feynman would revise before considering final. Little did I suspect that I was only postponing my task by ∼30 years.

Unfortunately most of those corrected drafts became dispersed with the other TAs, who have left physics. In my possession are seven lectures that I prepared for publication, at least some of which were revised by Feynman. (These are denoted by an asterisk * in the section headings.) As for the rest, I report what is in my class notes, trying to convey their intent as best I can. Based upon the rather extensive revisions he made to some of my first drafts, the sections he did not check are unlikely to do justice to all of his intended meanings. Certain parts call for elaboration, but I abstain from restoring longer explanations where I have no record of what Feynman actually said. These fully revised lectures can be found in sections 6, 7, 8, 9, 12. I was able to supplement my notes in some places with his own (mostly very sketchy) lecture notes, that are available from the Caltech Archives, Folder 41.7 of the Feynman Papers.

For the lectures of Jan. 5, 12 and 14, 1988, I was able to refer to tape recordings that were kindly provided by Arun K. Gupta, one of the former TAs. I have placed verbatim transcriptions of these lectures in the appendix, as a supplement to the more conventional versions in the main document. The quality of the recordings makes it impossible to reproduce every word, and ellipses indicate words or passages that I could not make out. This is especially the case toward the end of long explanations, where Feynman's voice would tend to diminish greatly, whereas at the beginning he might almost be shouting. These recordings are available alongside the lectures as supplementary material. I have preserved as much as possible his original words to convey the style of delivery, which was considerably more colorful and colloquial than the tone he adopted in the drafts to be published. The reader who compares these "raw" versions with the revised ones will understand why it was sometimes challenging to correctly capture Feynman's intended meanings.

One thing you may notice, and that struck me as an educator now myself, is that Feynman was never in a rush to explain anything (although at times he would speak very fast), nor did he eschew repeating himself, perhaps in several different ways, to try to get his point across. And of course there was his bent for telling stories, which I had forgotten about in the context of this course, since I had omitted them from my written notes. The "interlude," section 14, which were Feynman's remarks at the start of the new term, is kept in the main body of the text; it has a few interesting stories, and shows that he would make time to help a high school student with his geometry.

I have the impression that in some places Feynman had not prepared carefully and was working things out on the spot, sometimes getting them not quite right, and at times seemingly meandering through the material. This was apparent for example in the early lectures on QCD, where in subsequent class sessions he came back and revised previous equations to correct the details. It is interesting that no notes corresponding to the QCD lectures appear in the Caltech Archives folder, suggesting he was speaking extemporaneously. There is also repetition of already introduced material. Perhaps this was a deliberate pedagogical strategy, since it gave the students time to digest the concepts and to see it being derived from scratch. It is also possible that his terminal illness was interfering with his ability to prepare as well as he might have liked to. These detours would have been smoothed over in the version destined for publication, had there been time for him to revise the notes.

Although there were no homework assignments, there were some recommended problems that are included in the lectures. Moreover about a month before the end of the first term, when students were starting to think about the upcoming final exams, Feynman decided that each of them should do an original research project relating to QCD. I recall that many were dumbfounded when this announcement was made. Such an unexpected demand made by a lesser instructor would have created some outcry, but to a decree from the great man nobody objected, and everyone somehow managed to carry out the task: it was a privilege. Feynman of course graded the projects himself, and he comments on them in the interlude section.

One may wonder what the specific content of the un-

finished part of the course might have been. Feynman announces at the beginning of the second term that it will be half on perturbative methods followed by non-perturbative. At that time he was interested in QCD in 1+1 dimensions, as an exactly solvable model that might shed light on the real theory. He started working with a few graduate students on this subject, including Sandip Trivedi.

His private course notes reveal a different direction; around 20 of the 60 pages are devoted to reformulating vector spaces and calculus in arbitrary noninteger dimensions, which he discusses in lectures 20-21. His intent was to combine this with Schwinger's functional formulation of field theory, presented in lectures 21-22, to overcome the difficulty of defining the path integral in noninteger dimensions. Also in those notes is some material on chiral symmetry breaking by the axial anomaly and theta vacua in QCD, that he did not have time to present. No doubt the students would have been exposed to his ideas for deepening our understanding of the strong interactions, had he lived until the end of the course.

Feynman was an inspiring teacher, presenting everything in an incisive and fascinating way, that obviously had his own mark on it. He reinvented the subject as was his wont, even if he was not the first to discover, for example, the Fadeev-Popov procedure for gauge fixing the path integral. In the final meetings, he was too weak to stand at the board, and he delivered the lectures while seated. He died less than three weeks following the last lecture. His passion for transmitting the excitement of physics to a new generation never waned.

Sorry this took so long, professor.

James M. Cline
Montréal, 2020

1. THE QUARK MODEL (10-15-87)

We begin our exploration of the strong interactions with a survey of the hadronic particles, interpreted from the quark model perspective. The spin-1/2 baryons are arranged in an octet in the plane of mass versus charge, and likewise the spin-3/2 baryons form a decuplet, as shown in fig. 1. The quark content is indicated for the decuplet states, where the quarks u, d, s have charge $+2/3, -1/3, -1/3$ respectively, and we take the opposite convention for the sign of strangeness than is usual.

Detailed properties of the baryons can be understood within the quark model by constructing the flavor/spin wave functions for the states. Consider the Δ^0 state ($J = 3/2$), whose upper two spin states are given by

$$\Delta^0 : \begin{cases} ddu\ \uparrow\uparrow\uparrow, & J_z = 3/2 \\ ddu\ \frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow), & J_z = 1/2 \end{cases} \tag{1.1}$$

Compare this to the neutron and Σ^0 ($J = 1/2$),

$$\begin{aligned} N^0 &: \ ddu\ \tfrac{1}{\sqrt{6}}(-2\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow), \quad J_z = 1/2 \\ \Sigma^0 &: \ uds\ \tfrac{1}{\sqrt{6}}(-2\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow) \end{aligned} \tag{1.2}$$

For Δ^0 and N^0, the coefficients of the $\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow$ spin terms had to be equal, since they are symmetric under interchange of the first two quarks, which have identical flavors (dd). However this is not a constraint for the uds baryons, so there must exist an additional state Λ^0

$$\Lambda^0 : \ uds\ \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \tag{1.3}$$

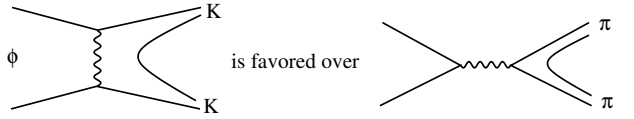
that has isospin 0. The fact that the mass eigenstates are also eigenstates of isospin indicates that u and d are approximately degenerate, compared to the scale of the hadron masses.

Similarly the mesons can be arranged into multiplets, as we illustrate for the $J = 1^-$ vector mesons in fig. 2. The wave functions are given by

$$\begin{aligned} \omega &: \ \tfrac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})\ \uparrow\uparrow \\ \rho^+ &: \ u\bar{d}\ \uparrow\uparrow; \quad \rho^0 : \ \tfrac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})\ \uparrow\uparrow; \quad \rho^- : \ d\bar{u}\ \uparrow\uparrow \\ K^{*-} &: \ s\bar{u}\ \uparrow\uparrow; \quad K^{*0} : \ s\bar{d}\ \uparrow\uparrow; \\ \overline{K^*}^+ &: \ u\bar{s}\ \uparrow\uparrow; \quad \overline{K^*}^0 : \ d\bar{s}\ \uparrow\uparrow; \\ \phi &: \ s\bar{s}\ \uparrow\uparrow \end{aligned} \tag{1.4}$$

It is interesting to notice that the ω and ρ^0 are very close to each other in mass. What do we learn about the strong interactions from this near-degeneracy? Apparently, the strong interactions conserve isospin.

It is also interesting to observe that ϕ decays much faster into KK than into pions. This is an example of Zweig's rule (OZI suppression), that can be pictured diagrammatically by the statement that



One might wonder whether OZI suppression in this example is somehow related to the degeneracy of the ϕ-ω system. In fact there is a connection: if ω had some $s\bar{s}$ content rather than being purely made from $u\bar{u}$ and $d\bar{d}$, which would spoil the degeneracy, then by the same mixing ϕ would also have light quark content, allowing for decays into pions without going through the annihilation diagram.

The pseudoscalar mesons ($J^P = 0^-$) have a different flavor structure from the vector mesons, apart from the similarities between the two isotriplets ρ and π,

$$[\pi^+,\ \pi^-,\ \pi^0] : \ \tfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\left[u\bar{d},\ d\bar{u},\ \tfrac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})\right] \tag{1.5}$$

In this case there is mixing between the isosinglets,

$$\begin{aligned} \eta\,(546) &\ \sim\ (u\bar{u} + d\bar{d}) - 1.4\, s\bar{s} \\ \eta'(960) &\ \sim\ (u\bar{u} + d\bar{d}) + 0.7\, s\bar{s} \end{aligned} \tag{1.6}$$

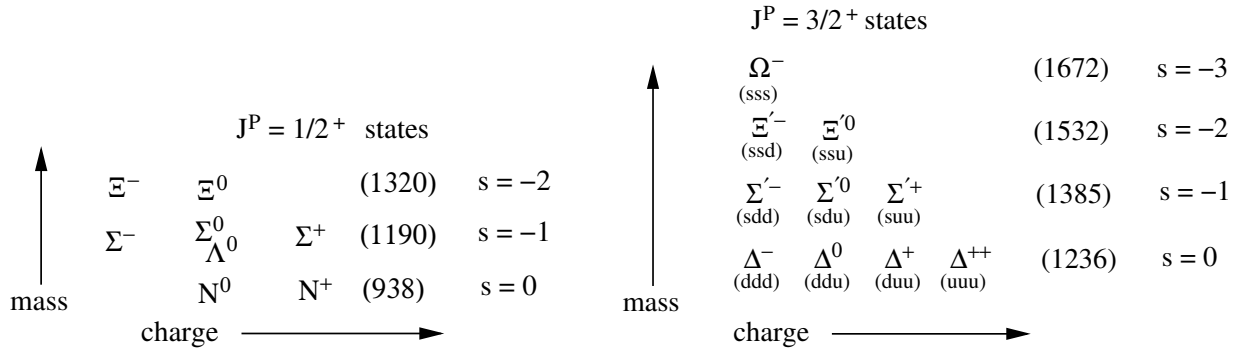


FIG. 1: The baryon octet (left) and decuplet (right). Masses indicated in MeV/c^2.

Why isn't η purely $(u\bar{u} + d\bar{d})$, in analogy to ω, which would have made it approximately degenerate with the pions? This has to do with chiral symmetry breaking, which is specific to QCD and not accounted for by the quark model.

An interesting prediction of the quark model is electromagnetic matrix elements, that determine the baryon magnetic moments. We consider those of the proton and the neutron, where the proton wave function is

$$|p\rangle = (uud)\,\tfrac{1}{\sqrt{6}}(-2\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow) \qquad (1.7)$$

The magnetic moment is given by

$$\left\langle p \left| \frac{q\hbar}{2m}\,\sigma_z \right| p \right\rangle \qquad (1.8)$$

where q is the charge operator acting on the quarks, and $m = m_p/3$ is the constituent quark mass. Using (1.7),

$$\begin{aligned} q\sigma_z|p\rangle = \frac{uud}{\sqrt{6}}\Big(&\frac{2e}{3}(-2\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \\ &+ \frac{2e}{3}(-2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow) \\ &- \frac{e}{3}(+2\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)\Big) \end{aligned} \qquad (1.9)$$

we find

$$\left\langle p \left| \frac{q\hbar}{2m}\,\sigma_z \right| p \right\rangle = 3\,\mu_N \qquad (1.10)$$

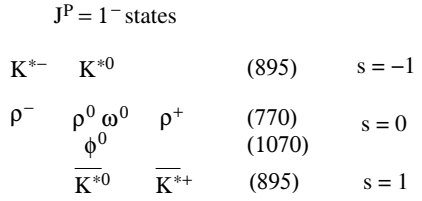


FIG. 2: The vector meson nonet.

where $\mu_N = e\hbar/2m_p$ is the nuclear magneton. The analogous calculation for the neutron (see eq. (1.2)) gives $-2\,\mu_N$. These predictions are compared to the measured values in the table 1.[1]

These predictions can be corrected, as shown in the third row of the table, by taking a more realistic value of the constituent u and d quark masses, $m_q = 1085/3 \cong 362\,\text{MeV}$ instead of $m_p/3$.[2] Further improvement might arise from taking into account isospin breaking; the u and d masses are not exactly the same. We must certainly take SU(3) flavor breaking into account for the s quark, whose constituent mass is $m_s = 1617/3 = 539\,\text{MeV}$. We can then predict the other magnetic moments as

$$\begin{aligned} \frac{q\sigma_z}{m}|\Lambda^0\rangle &= \frac{sud}{\sqrt{2}}\frac{q\sigma_z}{m}\,(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow) \to -\frac{1}{3m_s} \\ \frac{q\sigma_z}{m}|\Sigma^+\rangle &= \frac{sud}{\sqrt{6}}\frac{q\sigma_z}{m}\,(2\downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow) \\ &\to \tfrac{4}{6}\left[+\tfrac{1}{3m_s} + \tfrac{2}{3m_q} + \tfrac{2}{3m_q}\right] + \tfrac{1}{3}\left[-\tfrac{1}{3m_s}\right] \\ &= \tfrac{1}{9m_s} + \tfrac{8}{9m_q} \end{aligned} \qquad (1.11)$$

	p	n	Λ	Σ^+	Σ^-	Ξ^-
$\mu_{\text{th.}}$	3	−2		3	−1	−1
$\mu_{\text{exp.}}$	2.79	−1.93	-0.61±.03	2.83±.25	-1.48±.37	-1.85±.75
$\mu_{\text{corr.}}$	2.59	−1.73	−0.58	2.50	−0.96	−0.49

TABLE I: Predicted and observed baryon magnetic moments, in the quark model.

[1] In class, RPF only presented the n and p values, and omitted the "corrected" predictions. I have restored these and some of the related discussion from his private notes.

[2] It is not explained in his notes where the number 1085 comes from; probably it is a consequence of taking the spin-spin interactions into account in the baryon mass calculation.

$$\begin{aligned}
\frac{q\sigma_z}{m}|\Sigma^-\rangle &= \frac{sdd}{\sqrt{6}}\frac{q\sigma_z}{m}(2\downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow) \\
&\to \tfrac{4}{6}\left[+\tfrac{1}{3m_s} - \tfrac{1}{3m_q} - \tfrac{1}{3m_q}\right] + \tfrac{1}{3}\left[-\tfrac{1}{3m_s}\right] \\
&= \tfrac{1}{9m_s} - \tfrac{4}{9m_q} \\
\frac{q\sigma_z}{m}|\Xi^-\rangle &= \frac{dss}{\sqrt{6}}\frac{q\sigma_z}{m}(2\downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow) \\
&\to \tfrac{1}{9m_q} - \tfrac{4}{9m_s} \qquad (1.12)
\end{aligned}$$

These are in rather good agreement with the data, except for the Ξ^-.

The nonrelativistic quark model can also be used to predict the axial vector current matrix elements, $\sigma_z\gamma^5$. In the quark model we find that $\sigma_z\gamma^5$ gives $+1$ for u and -1 for d, leading to the prediction

$$g_A = \tfrac{4}{6}[1+1+1] + \tfrac{1}{3}[-1] = 5/3 \qquad (1.13)$$

for the proton,[3] which is high compared to the experimental value 1.253 ± 0.007.

Exercise. What kind of baryon states do you expect when there is one unit of internal angular momentum?

2. OTHER PHENOMENOLOGICAL MODELS (10-20-87)

There are complementary phenomenological models for describing the strong interactions, which we briefly review here. The first is the relativistic string, which was inspired by the observed Regge trajectories. These are plots of the spin versus mass squared of hadronic resonances, as in fig. 3. One considers only families of resonances having the same parity, requiring J to jump by $\Delta J = 2$. Empirically the trajectory is linear, which was not predicted by the quark model. However the relativistic string, illustrated in fig. 4, gets the correct relation.[4] In the simplest version of the model, the masses

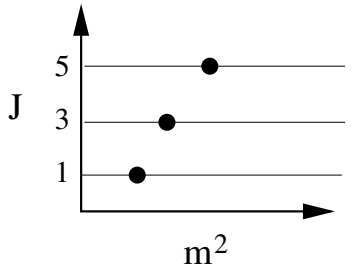


FIG. 3: A Regge trajectory.

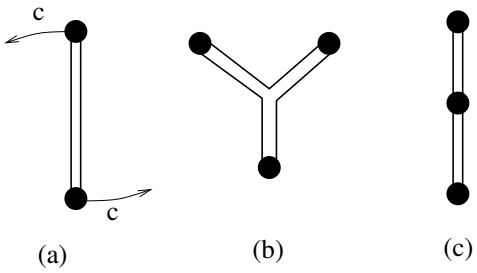


FIG. 4: String model for mesons (a) and baryons (b,c).

of quarks or antiquarks on the ends of the string are neglected, and one cares only about the constant tension T = energy/length of the strings, which represent flux tubes of the strong interaction field. One finds that m^2 (the energy squared) is proportional to the angular momentum, with the endpoints of the string moving at the speed of light. To explain the linear Regge trajectories of baryons in this picture, one could imagine flux tube configurations as in fig. 4(b,c). Configuration (c) would obviously lead to the same prediction of linear Regge trajectories as for mesons.

In the bag model (fig. 5(a)), quarks in a hadron "push away the vacuum" and move around in this evacuated region with nearly zero mass. It takes energy to make the hole, which is interpreted as the hadron mass. An application is to the decay $\rho \to e^+e^-$ (fig. 5(b)), where we recall that $\rho = (u\bar{u} - d\bar{d})/\sqrt{2}$. One must know the wave function of the $q\bar{q}$ bound state at the origin, $\psi(0)$, to estimate the amplitude. Although the bag model gives a reasonable estimate for $\psi(0)$, the bag nevertheless turns out to be too stable to get the rate right. One needs to make it more dynamical than in the bag model picture, so that it shrinks more easily when the q and $\bar{q}$ are close to each other. And the bag should turn into a flux tube when the pair is well-separated.

The parton model is useful for describing high-energy processes, including inelastic scattering of electrons on nucleons. An example is shown in fig. 6, for the case of $eN \to eN\pi$. Let us think about the partons in the initial state nucleon, in a reference frame where it is moving to the right with 4-momentum $p^\mu = (E, p, 0, 0)$ and $E \sim 10\,\mathrm{GeV}$ for example, and the virtual photon 4-momentum is $q^\mu = (0, \vec{Q})$. A parton in the nucleus will

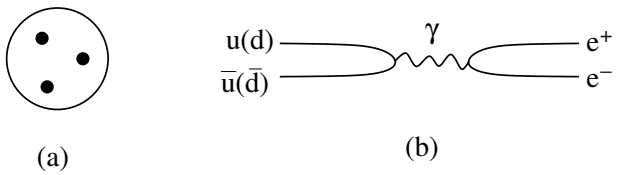


FIG. 5: (a) Bag model for baryons. (b) Electromagnetic decay of ρ meson.

[3] This calculation, also taken from his written notes, seems to be based on unstated insights from the extended quark model analysis that takes into account the small components of the Dirac spinors.

[4] This is worked out in the next lecture.

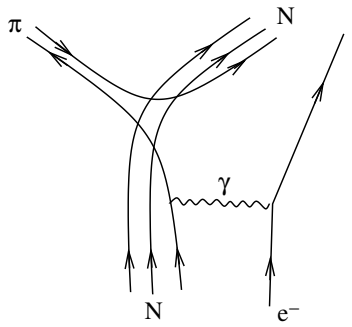


FIG. 6: High-energy electron-nucleon scattering

have momentum components

$$p_\parallel = xp, \quad p_\perp \sim 300\,\mathrm{MeV} \tag{2.1}$$

parallel and perpendicular to the beam, respectively, where x is the momentum fraction of the parton that interacts with the virtual photon. In this frame, its momentum just gets reversed after scattering, $px \to -px = Q/2$, since its energy changes by $q^0 = 0$. In an arbitrary frame we can write the momentum fraction as

$$x = \frac{Q^2}{2pQ} = \frac{-q^2}{-2p\cdot q}\,. \tag{2.2}$$

The momentum distribution of partons in the nucleon can be thought of as coming from their respective wave functions, written in momentum space. Naively, we would expect the probability to find a quark with momentum in the interval [5, 5.5] GeV in a 10 GeV proton to be the same as for the interval [10, 11] GeV in a 20 GeV proton. Each parton has its own probability distribution

$$\begin{aligned} u(x),\, d(x),\, s(x): &\qquad \text{quarks} \\ \bar u(x),\, \bar d(x),\, \bar s(x): &\qquad \text{antiquarks} \\ g(x): &\qquad \text{gluon} \end{aligned} \tag{2.3}$$

For example $u(x)$ is the probability density for finding a u quark with momentum fraction x. Of course these definitions depend upon which hadron the parton belongs to. If we define the above functions as belonging to the proton, then the amplitude for the photoproduction process is proportional to

$$\tfrac{4}{9}u(x)+\tfrac{1}{9}d(x)+\tfrac{1}{9}s(x)+\tfrac{4}{9}\bar u(x)+\cdots, \quad (ep\to eN\pi) \tag{2.4}$$

for scattering on protons, whereas it is

$$\tfrac{4}{9}d(x)+\tfrac{1}{9}u(x)+\tfrac{1}{9}s(x)+\tfrac{4}{9}\bar u(x)+\cdots, \quad (en\to eN\pi) \tag{2.5}$$

for scattering on neutrons since $u(x)$ in a proton must be equal to $d(x)$ in a neutron. The amplitudes will of course also depend upon Q.

Challenge. Compute the width for $\phi \to e^+e^-$.

3. DEEP INELASTIC SCATTERING; ELECTRON-POSITRON ANNIHILATION (10-22-87)

In the last lecture we saw that the electron-proton scattering cross section is proportional to a function

$$F^{ep}(x) = \tfrac{4}{9}(u+\bar u) + \tfrac{1}{9}(d+\bar d) + \tfrac{1}{9}(s+\bar s) \tag{3.1}$$

where $u(x)$ is the probability density for finding a u quark with momentum fraction x in the proton. The momentum distribution functions are subject to constraints

$$\begin{aligned} 1 &= \begin{pmatrix}\text{proton}\\ \text{charge}\end{pmatrix} = \int \left[\tfrac{2}{3}(u-\bar u)-\tfrac{1}{3}(d-\bar d)-\tfrac{1}{3}(s-\bar s)\right] dx \\ 0 &= \begin{pmatrix}\text{neutron}\\ \text{charge}\end{pmatrix} = \int \left[\tfrac{2}{3}(d-\bar d)-\tfrac{1}{3}(u-\bar u)-\tfrac{1}{3}(s-\bar s)\right] dx \\ 0 &= \text{nucleon strangeness} = \int (s-\bar s)\, dx \end{aligned} \tag{3.2}$$

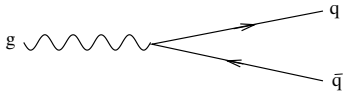
where again we assumed the neutron is related to the proton by interchange $u \leftrightarrow d$. From these it follows that

$$\begin{aligned} \int (u-\bar u)\, dx &= 2 \\ \int (d-\bar d)\, dx &= 1 \\ \int (s-\bar s)\, dx &= = 0 \end{aligned} \tag{3.3}$$

It has been shown that as $x \to 0$, the distribution functions have the behavior

$$u = \bar u = d = \bar d \sim \frac{0.24}{x} \tag{3.4}$$

Likewise, $s = \bar s$ scales as $1/x$. This behavior can be understood as coming from brehmsstrahlung of soft gluons, which have a distribution of $1/x$. These (virtual) gluons decay into soft quark-antiquark pairs,



explaining why all flavors have the same $1/x$ dependence at low x, regardless of whether they are particles or antiparticles: gluons can decay into all flavors equally. The gluons are known to comprise a significant fraction of the total partons,

$$\int_0^1 dx\, g(x) \approx 0.44 \tag{3.5}$$

So far, we have taken for granted that we know the charges of the quarks. One experiment that constrains the charges is annihilation of electrons and positrons, fig. 7. Denoting the cross section for annihilation into e^+e^- by $\sigma_{\rm el}$, we can express that for annihilation into $u\bar u$

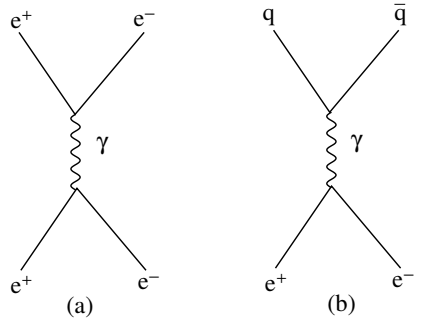


FIG. 7: Electron-positron annihilation into e^+e^- (a) and $q\bar{q}$ (b).

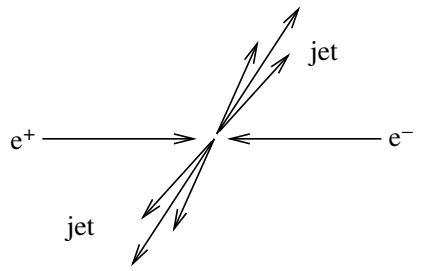


FIG. 8: Electron-positron annihilation into hadronic jets.

as[5]

$$\sigma_{u\bar{u}} \sim \frac{4}{9}\,\sigma_{\rm el}\cdot 3 \tag{3.6}$$

where the final factor of 3 is for the number of colors, which we must determine by some independent means. We have assumed here that final state interactions can be neglected. Similarly for $\sigma_{d\bar{d}}$ or $\sigma_{s\bar{s}}$ we get $\frac{1}{9}\,\sigma_{\rm el}\cdot 3$. For the inclusive cross section to produce hadrons, we add the three flavors together to obtain

$$\frac{\sigma_{e^+e^-\to\rm hadrons}}{\sigma_{\rm el}} = \frac{4}{3}+\frac{1}{3}+\frac{1}{3} = 2 \tag{3.7}$$

assuming the energy is below the c quark threshold. Above this threshold, $2 \to 10/3$, and above the b quark threshold $10/3 \to 11/3$.

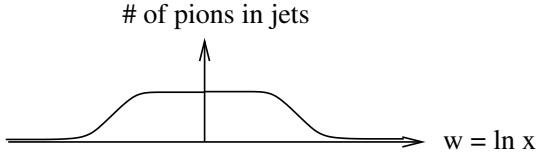


FIG. 9: Distribution of pions with momentum fraction x in jets.

[5] RPF has apparently ignored the t-channel contribution to $\sigma_{\rm el}$ here.

Or course what we really see is not quarks in the final state, but rather jets (fig. 8), primarily K's and π's, with smaller admixtures of nucleons and antinucleons. One can define probability distribution functions for hadrons in the jets in analogy to those of the partons, for example $\pi(x)$, where now the momentum fraction is defined as

$$x = \frac{\text{energy of } \pi}{\text{energy of } e^+e^-} \tag{3.8}$$

Like for the quarks, these distributions go like $1/x$ at small x. Denote the components of the π momentum parallel to and transverse to the average jet momentum as $p_\parallel$ and $p_\perp$. Then

$$\frac{dx}{x} = \frac{dp_\parallel}{E} = \frac{dp_\parallel}{\sqrt{m_\pi^2+p_\perp^2+p_\parallel^2}} \tag{3.9}$$

This shows that at small x the distribution of particles with momentum fraction x is flat as a function of $w = \ln x$, fig. 9. Lorentz transforming to a frame where the average momentum of the two jets does not add to zero causes the distribution to be translated to the right or left in w. More generally, we can define *fragmentation functions* $D_h^q(x)$ that denote the probability distribution for producing a hadron of type h and momentum fraction x from a jet originating from a quark of flavor q. These can be measured in deep inelastic scattering experiments.

The formation of two jets from the breaking of a string of strong interaction flux is in some ways analogous to a simpler problem, the spontaneous emission of an electron-positron pair from a constant electric field. Solving the Dirac equation in this background, I find that the probability of pair production is $\exp(-\frac{e^2}{8\pi}\,\frac{m^2+\vec{p}^{\,2}}{\vec{E}^2})$. Now imagine that the electric field is created by two charged plates that are moving together with velocity v. The pairs should be produced with net total momentum. But locally they are created from a uniform field, so how do they know they should have net momentum?

A related problem concerns the wave function of quarks in a stationary proton versus a moving proton. The wave function is not a relativistic invariant, nor even something that transforms nicely.

Exercise. From the Schrödinger equation, how does the solution for the wave function ψ transform when the potential changes by the Galilean transformation $V(\vec{r}) \to V(\vec{r}-\vec{v}t)$? (Answer: $\psi \to e^{im(\vec{v}\cdot\vec{r}-\frac{1}{2}\vec{v}^{\,2}t)}\psi(\vec{r}-\vec{v}t,t)$.)

Similarly, the wave function for positronium, $\psi(\vec{x}_1,\vec{x}_2,t)$, depending on the positions of its constituents, changes in a complicated way, as can be understood by Lorentz transforming and noting that in the new frame, the events that were $(t,\vec{x}_1)$ and $(t,\vec{x}_2)$ in the original frame are no longer simultaneous; see fig. 10. We have to evolve one particle forward and the other backward in time to find the new wave function. Hence ψ in the new frame, call it ψ', is not just a function of the original ψ, but rather $\psi' = f(\psi,H)$, depending also on the Hamiltonian H of the system.

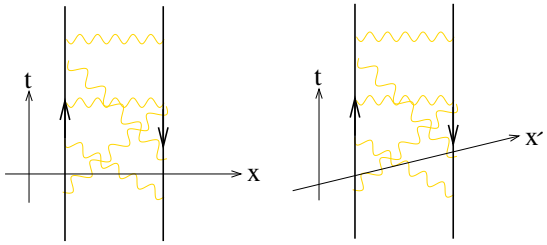


FIG. 10: Relativity of simultaneity for positronium consituents in the rest frame versus a boosted frame.

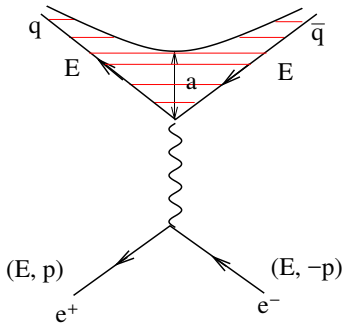


FIG. 11: Hadronization after $e^+e^- \to q\bar{q}$ production, showing the breaking of the QCD string.

We can say something more quantitative about the distributions of transverse and longitudinal momenta however. The center of mass energy of the e^+e is $2E \cong 2|\vec{p}|$. The produced quarks carry less momentum because of the mass produced in the QCD string during hadronization, $E = \sqrt{p_\parallel^2 + p_\perp^2 + m^2} = xp\sqrt{1 + (p_\perp^2 + m^2)/(xp)^2}$ (recalling that $p_\parallel = xp$). Hence[6]

$$E - p_\parallel \sim \frac{\#}{xp}\,. \tag{3.10}$$

A measure of momentum loss is given by the sum over the different kinds of hadronic particles h produced,

$$\begin{aligned} |\Delta\vec{p}| &= \sum_h \int (E - p_\parallel)\frac{dp_\parallel}{E} C_h\, d^2p_\perp \\ &= \sum_h \int d^2p_\perp\, C_h\, (p_\parallel - E)\Big|_0^p \\ &= \sum_h \int d^2p_\perp\, C_h\, \sqrt{p_\perp^2 + m_h^2} \end{aligned} \tag{3.11}$$

where C_h depends on the particle. As shown, one can do the integral over $p_\parallel$ exactly. This gives us a way of measuring the string tension T, since the loss of momentum is given by Newton's law,

$$\sum_{L+R} |\Delta\vec{p}| = T \times a \tag{3.12}$$

where the sum is over both left and right sides of the diagram (the two jets) and a is the time it takes to break the string, as shown in fig. 11.

The theory of the string tension is not very quantitative. It comes from Regge trajectories, but these are not known for highly stretched strings, and moreover in the real situation there are quarks at the ends of the strings, that have not been taken into account.

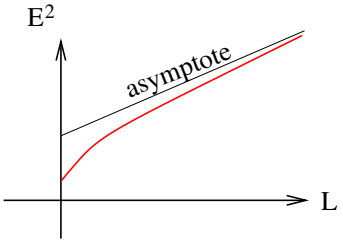


FIG. 12: Regge trajectory for the flexible bag model.

Exercise. The "flexible bag" model takes the quark mass to depend on the quark separation in a hadron, with Hamiltonian $H = \frac{1}{2}m(r)\frac{d}{dt}\vec{r}\cdot\frac{d}{dt}\vec{r} + V(r)$, $V(r) = kr$, $m(r) = \mu r$. The quark mass represents the inertia of the gluon field (bag). Prove that $E^2 \sim aL + b + c/L + \ldots$ for large angular momentum L. Find E for $L = 0$ and for large L.[7] [RPF shows graphically his result in fig. 12.]

Problem. What happens for the relativistic treatment of the string? We must formulate a relativistic equation. String theory![8]

Solution: The proper tension T is the energy per unit length. The radial variable goes from 0 to a, so the velocity varies as $v(r) = r/a$ along the string, which rotates at angular frequency $\omega = 1/a$. The differential force acting on an element of the string is $dF = \omega\mu v dr$ where $\mu = T/\sqrt{1 - v^2}$. Therefore the total energy and angular momenta are

$$\begin{aligned} E &= 2\int_0^a \frac{T\,dr}{\sqrt{1-v^2}} = 2Ta\int_0^1 \frac{dv}{\sqrt{1-v^2}} = \pi Ta \\ J &= 2\int_0^a \mu v r\,dr = 2Ta^2\int_0^1 \frac{v^2\,dv}{\sqrt{1-v^2}} = \tfrac{\pi}{2}Ta^2 \end{aligned}$$

and we understand the Regge trajectory behavior, $E^2 = 2\pi TJ$. Comparing to data, $2\pi T = 1.05\,\mathrm{GeV}^2$, giving $T = 0.167\,\mathrm{GeV}^2$.

[6] This equation which holds at large xp does not seem to be needed for what follows.

[7] RPF in his private notes devotes four pages to working this out, first classically for circular orbits, then quantum mechanically using exponential and Gaussian variational anzätze for the wave function.

[8] This was around the time of the first string revolution. I reproduce the following answer from RPF's private notes.

References.[9]

Quark model references:

O. W. Greenberg, "Spin and Unitary Spin Independence in a Paraquark Model of Baryons and Mesons," Phys. Rev. Lett. **13**, 598 (1964). doi:10.1103/PhysRevLett.13.598

O. W. Greenberg and M. Resnikoff, "Symmetric Quark Model of Baryon Resonances," Phys. Rev. **163**, 1844 (1967). doi:10.1103/PhysRev.163.1844

R. P. Feynman, M. Kislinger and F. Ravndal, "Current matrix elements from a relativistic quark model," Phys. Rev. D **3**, 2706 (1971). doi:10.1103/PhysRevD.3.2706

Isgur[10]

MIT bag model references:

A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn and V. F. Weisskopf, "A New Extended Model of Hadrons," Phys. Rev. D **9**, 3471 (1974). doi:10.1103/PhysRevD.9.3471

T. A. DeGrand, R. L. Jaffe, K. Johnson and J. E. Kiskis, "Masses and Other Parameters of the Light Hadrons," Phys. Rev. D **12**, 2060 (1975). doi:10.1103/PhysRevD.12.2060

4. QUANTUM CHROMODYNAMICS (10-27-87)

We will denote color indices by $a, b, \cdots = r, b, g$ and flavor by f so that the noninteracting part of the quark Lagrangian is[11]

$$\sum_f \bar\psi^f_{\bar a}(i\partial\!\!\!/ - \mu_f)\psi^f_a \tag{4.1}$$

while the interaction term (for a single flavor) is

$$\bar\psi_{\bar a}\, A^\mu_{\bar a b}\gamma_\mu\psi_b \tag{4.2}$$

and the gluon kinetic term is[12]

$$\mathrm{tr}(\partial_\mu A^{b\bar a}_\nu - \partial_\nu A^{b\bar a}_\mu)^2 \equiv \mathrm{tr}(E^{b\bar a}_{\mu\nu})^2 \tag{4.3}$$

(Here only the noninteracting part of the field strength is used.) The gauge transformations are given by

$$\begin{aligned} \psi' &= \Lambda\psi \\ A'_\mu &= \Lambda^\dagger A_\mu \Lambda + \Lambda^\dagger i\partial_\mu\Lambda \\ E_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] \\ &\to \Lambda^\dagger E_{\mu\nu}\Lambda = E'_{\mu\nu} \end{aligned} \tag{4.4}$$

The fully gauge invariant QCD Lagrangian is

$$\begin{aligned} \mathcal{L}_{QCD} &= \sum_f \bar\psi_f[i\partial\!\!\!/ - A\!\!\!/ - \mu_f]\psi_f \\ &+ \frac{1}{2g^2}\mathrm{tr}\, E_{\mu\nu}E_{\mu\nu} \end{aligned} \tag{4.5}$$

The quark-gluon interaction can also be written as

$$\mathcal{L}_{q-\mathrm{int}} = \mathrm{tr}\, A_\mu J_\mu \tag{4.6}$$

with the current

$$J_\mu = \sum_f J_{\mu,f}; \qquad J^{\bar b a}_{\mu,f} = \bar\psi^{\bar b}_f\gamma_\mu\psi^a_f \tag{4.7}$$

Notice that there are $6\times4\times3$ (quark) and 4×8 (gluon) field degrees of freedom at each point in spacetime.

Exercises. 1. If $B(x)$ is a 3×3 matrix transforming as $B' = \Lambda^\dagger B\Lambda$, show that $\partial_\mu B$ does not transform homogeneously in this way, but $\partial_\mu B - i[A_\mu, B]$ does. *I.e.,* $\partial_\mu - i[A_\mu,\ \ ] \equiv D_\mu$ is the covariant derivative for fields transforming in the octet representation.

2. Prove that $[D_\mu, D_\nu]B = i[E_{\mu\nu}, B]$.

3. If $A_\mu \to A_\mu + \delta A_\mu$ then $\delta E_{\mu\nu} = D_\mu\delta A_\nu - D_\nu\delta A_\mu$.

4. By varying A_μ in $\mathcal{L}$, show that $D_\mu E_{\mu\nu} = g^2 J_\nu$, where J is the quark current.

5. Derive the equation of motion of the quarks, $(i\partial\!\!\!/ - \mu_f - A\!\!\!/)\psi = 0$.

6. From this, show that $D_\mu J_{\mu,f} = 0$; also $\partial_\mu \mathrm{tr}\, J_{\mu,f} = 0$.

7. Show that #4 is meaningless unless $D_\mu J_\mu = \sum_f D_\mu J_{\mu,f} =$. Hint: #2.

8. Show (identically, not as a consequence of the equations of motion but due to the form of $E_{\mu\nu}$) that $D_\mu E_{\nu\sigma} + D_\nu E_{\sigma\mu} + D_\sigma E_{\mu\nu} = 0$. Note that if $\tilde E_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\sigma\tau}E_{\sigma\tau}$ then this can be written as $D_\mu\tilde E_{\mu\nu} = 0$.

9. Define color electric and magnetic fields $E_x = E_{xt}$ *etc.* and $B_z = E_{xy}$ *etc.* (check the signs). Then rewrite the field equations in terms of these quantities. Let $\mathbb{E} = (E_x, E_y, E_z)$ *etc.* Show that

$$\begin{aligned} \mathbb{D}\cdot\mathbb{B} &= 0 \\ \mathbb{D}\times\mathbb{E} + D_t\mathbb{B} &= 0 \\ \mathbb{D}\cdot\mathbb{E} &= g^2\rho \text{ where } \rho = J_t \\ \mathbb{D}\times\mathbb{B} - D_t\mathbb{E} &= g^2\mathbb{J} \end{aligned} \tag{4.8}$$

where $\mathbb{D} = \mathbb{V} - i[\mathbb{A},\ \ ]$ and $D_t = \partial_t - i[A_t,\ \ ]$.

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[9] These were originally given at the end of lecture 5, but logically they belong here since they pertain to quark models.

[10] No specific references are given, but probably RPF had in mind Isgur's papers from 1978-1979 on the quark model.

[11] RPF omits coupling constants and numerical factors in eqs. (4.2-4.4), but restores them in (4.5).

[12] RPF uses $E_{\mu\nu}$ and $F_{\mu\nu}$ interchangeably for the field strength.

Define the matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix} \quad \lambda_8 = \sqrt{\tfrac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\lambda_0 = \sqrt{\tfrac{2}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.9}$$

Our convention is that $\mathrm{tr}(\lambda_i \lambda_j) = 2\delta_{ij}$. We define $A_\mu = \frac{1}{2} A^a_\mu \lambda_a$. There is no A^0_μ term because this would correspond to an extra U(1) force. Let $\vec{A}_\mu = (A^1_\mu, \ldots, A^8_\mu)$.

Exercises, continued. 10. Show that $\Lambda^\dagger \partial_\mu \Lambda$ is traceless, *i.e.,* it has no λ_0 component.

The structure constants f_{ijk} are defined through

$$\left[(\tfrac{1}{2}\lambda_i), (\tfrac{1}{2}\lambda_j)\right] = i f_{ijk} \, (\tfrac{1}{2}\lambda_k) \tag{4.10}$$

One can show that f_{ijk} is totally antisymmetric. We also define the anticommutator

$$\{\lambda_i, \lambda_j\} = 2\, d_{ijk}\, \lambda_k \tag{4.11}$$

Unlike f_{ijk}, we can find the extra generator $k = 0$ amongst those on the right-hand side.[13]

11. Show that $\partial_\mu A^i_\nu - \partial_\nu A^i_\mu - f_{ijk} A^j_\mu A^k_\nu = E^i_{\mu\nu}$. We can rewrite this as $\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu - \vec{A}_\mu \times \vec{A}_\nu = \vec{E}_{\mu\nu}$ by defining a cross product in color space as

$$(\vec{C} \times \vec{D})_i = f_{ijk}\, C_j\, D_k \tag{4.12}$$

Similarly define the dot product

$$\vec{C} \cdot \vec{D} = \sum_i C_i D_i \tag{4.13}$$

12. Prove that

$$\vec{C} \cdot (\vec{C} \times \vec{D}) = 0$$

and

$$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0$$

However (you don't need to prove this), the familiar identity $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$ is only true for SU(2) and not for general SU(N).

13. Show that

$$C = D_\mu B \implies \vec{C} = \partial_\mu \vec{B} - \vec{A}_\mu \cdot \vec{B} \tag{4.14}$$

14. Rewrite $\mathcal{L}$ using component notation.

[13] This seems to be a notational innovation of RPF.

4.1. Geometry of color space

Consider successive transformations $\psi' = \Lambda\Psi$, $\psi'' = M\psi'$. Then $\psi'' = N\psi \equiv M\Lambda\psi$ obviously. This is an example of the group multiplication law for the color rotations. For many purposes we may be interested in infinitesimal rotations, $\Lambda = 1 + i\mathbf{a}$. Under this, the gauge field transforms as

$$\begin{aligned} A'_\mu &= (1 - i\mathbf{a}) A_\mu (1 + i\mathbf{a}) + (1 - i\mathbf{a}) \partial_\mu (1 + i\mathbf{a}) \\ &= A_\mu - i[\mathbf{a}, A_\mu] + i\partial_\mu \mathbf{a} \\ &= A_\mu + i D_\mu \mathbf{a} \end{aligned} \tag{4.15}$$

So it is always possible to impose temporal gauge, $A_0 = 0$, since this only requires solving a first order differential equation. In the following however we will discuss a difficulty that arises when charges are present.

What happens to a quark's color as it is transported through a gluon field? The transformation between two sets of color axes separated by a distance Δx_μ can be written as $1 + iA_\mu \, \Delta x_\mu$. Now suppose that every set of axes is changed locally by a rotation $\Lambda(x)$. Then the new transformation relating the two sets of axes is

$$\Lambda^\dagger(x + \Delta x)(1 + iA_\mu \, \Delta x_\mu)\Lambda(x) \equiv 1 + iA'_\mu \, \Delta x_\mu \tag{4.16}$$

Therefore

$$A'_\mu = \Lambda^\dagger A_\mu \Lambda + \Lambda^\dagger i \partial_\mu \Lambda \tag{4.17}$$

which is the finite version of (4.15).

4.2. Quark-antiquark potential

It would be very satisfying if we could justify some of the phenomenological approaches I considered earlier, using QCD as a starting point. Heavy quarkonium systems, being approximately nonrelativistic, are the simplest systems to consider, and can be described by a potential of the form

$$V \sim \frac{\alpha}{r} + br + \ldots \tag{4.18}$$

where I have omitted the spin-spin and spin-orbit interactions. (In the complete Hamiltonian there is also an annihilation term H_A that can cause transitions like $u\bar{u} \leftrightarrow s\bar{s}$, that give rise to η-η' mixing.) The terms written describe the linearly confining potential representing the mass of the string connecting the quark to the antiquark, and the Coulomb-like interaction, which might rather be something like $e^{-\mu r}/r$.

One can also make predictions for relativistic systems like the vector mesons; see S. Godfrey, N. Isgur, Phys. Rev. D32, 189 (1985). Then it is advantageous to use harmonic oscillator wave functions as a basis for computing matrix elements of the Hamiltonian to get a good approximate solution and compare to the data. Not only

can one compute the mass spectrum, but also strong interaction decay amplitudes, such as for $\phi \to K\bar{K}$. But all of this still relies on making a reasonable guess for the form of the potential, and it would be preferable to derive these interactions directly from QCD.

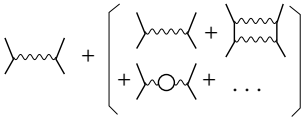
Let us recall how the analogous calculation works in QED, for the potential between a proton and an electron. We start with the fundamental interactions,

$$\mathcal{L} = \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi}_e \not{D} \psi_e + \bar{\psi}_p \not{D} \psi_p \tag{4.19}$$

and from this we would like to derive the nonrelativistic effective Hamiltonian

$$H_{NR} = \frac{p^2}{2\mu} + V(r) \tag{4.20}$$

The potential can be computed perturbatively, by Fourier transforming the amplitude,

$V(r) \sim$  (4.21)

But for QCD we know that the linear term is a nonperturbative effect, so a different approach is needed.

A better way might be to solve the classical equations of motion for the gauge field, in the presence of a source term, where the Lagrangian is

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} + A_\mu J_\mu \\ &= \tfrac{1}{2} \left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu \right) + A_\mu J_\mu \\ &\to -\tfrac{1}{2} \left(A_\nu \Box A_\nu + A_\nu \partial_\mu \partial_\nu A_\mu \right) + A_\mu J_\mu \end{aligned} \tag{4.22}$$

This gives the equation of motion

$$-\Box A_\mu + \partial_\mu \partial_\nu A_\nu + J_\mu = 0 \tag{4.23}$$

which by taking the divergence implies the current is conserved, $\partial_\mu J_\mu = 0$. Since $\partial \cdot F = \partial_\mu F_{\mu\nu} = \partial_\mu (\partial_\mu A_\nu - \partial_\nu A_\mu)$, eq. (4.23) can be written in the gauge-invariant form

$$\partial_\nu F_{\mu\nu} = J_\mu \tag{4.24}$$

Now we must solve (4.24) when the source is $J_0 = e\left(\delta(\vec{r}) - \delta(\vec{r} - \vec{a})\right)$, supposing that the two charges are located at the origin and at $\vec{r} = \vec{a}$ respectively. In electrodynamics this is easy, thanks to the linearity of the theory. We just superpose the solutions from the two sources, call them

$$\vec{E}_1 = \frac{q_1}{r^2}\, \hat{r}, \quad \vec{E}_2 = \frac{q_2}{r'^2}\, \hat{r}' \tag{4.25}$$

where $\vec{r}' = \vec{r} - \vec{a}$. Then we can compute the interaction energy by integrating the energy density in the fields, $\mathcal{E} \sim |\vec{E}_1 + \vec{E}_2|^2$:

$$V(a) = \int d^3x\, \mathcal{E}_{\rm int} = \int d^3x\, 2\vec{E}_1 \cdot \vec{E}_2 \tag{4.26}$$

This shows how one might be able to compute the quark-antiquark potential without relying on perturbation theory; it would require knowing the classical solution for the gluons fields in the presence of a static source.

4.3. Classical solutions

Therefore we would like to solve for the chromoelectric field in the presence of a source. However it is no longer possible write this only in terms of the field strength, as we could for QED. In the temporal gauge $A_0 = 0$, the Gauss's law equation is

$$\mathbb{D} \cdot \mathbb{E} = \mathbb{\nabla} \cdot \mathbb{E} - i \left(\mathbb{A} \cdot \mathbb{E} - \mathbb{E} \cdot \mathbb{A} \right) = g^2 \rho \tag{4.27}$$

where

$$\rho = J_0 = \sum_f \bar{\psi}^b_f \gamma_0 \psi^a_f \tag{4.28}$$

which is a matrix in color space. Before fixing the gauge,

$$\mathbb{D} \cdot \mathbb{E} = \mathbb{D} \cdot (\mathbb{\nabla} A_0 - \partial_0 \mathbb{A} - [\mathbb{A}, A_0]) \to -\mathbb{D} \cdot \partial_0 \mathbb{A} \tag{4.29}$$

so another way of writing (4.27) in $A_0 = 0$ gauge is

$$-\mathbb{D} \cdot \partial_0 \mathbb{A} = g^2 \rho \tag{4.30}$$

However we should first verify that there is no obstacle to transforming to the $A_0 = 0$ gauge when external charges are present. An issue, as I will show, is whether one can consider the source to be static. Starting from some configuration with $A_0 \neq 0$, we would like to construct the gauge transformation that makes $A'_0 = 0$, by solving eq. (4.17) with $\mu = 0$. One can guess that it is a time-ordered exponential,

$$\Lambda = P \exp\left(i \int_{t_0}^{t} dt' A_0(t') \right) \tag{4.31}$$

and verify that this is a solution, since

$$\begin{aligned} \partial_0 \Lambda &= \partial_0 \left(1 + i \int_{t_0}^{t} dt' A_0(t') \right. \\ &\quad \left. - \int_{t_0}^{t} dt_1 A_0(t_1) \int_{t_0}^{t_1} dt_2 A_0(t_2) + \dots \right) \\ &= A_0(t) - A_0(t) \int_{t_0}^{t} dt' A_0(t') + \dots \end{aligned} \tag{4.32}$$

So there is no difficulty in transforming to the temporal gauge. But we must also consider how the source (4.23) transforms:

$$\rho \to \Lambda^\dagger \rho \Lambda \tag{4.33}$$

Recall that the quark changes its color when it emits a gluon; that's why the charge is a matrix. Does the gauge dependence mean that it makes no sense to ask what is the potential between two spatially separated charge matrices?

Before getting too ambitious and trying to solve with the source having both a quark and an antiquark, let's first imagine the seemingly easier case of a single quark, even though the solution is not expected to fall off at

large distances. Suppose that a quark starts out being red. At a later time, after emitting or absorbing a gluon, it is some linear combination of red, green, blue:

$$q(t) = \begin{pmatrix} r(t) \\ b(t) \\ g(t) \end{pmatrix} \tag{4.34}$$

where $|r|^2 + |b|^2 + |g|^2 = 1$, say. It gives a color charge matrix of the form $\rho_{\bar{a}b} = \bar{q}^{\bar{a}}\gamma_0 q^b$. Clearly a nontrivial solution will have time dependence, associated with the fact that the source is not a color singlet. To avoid this, we would have to include the antiquark contribution, so as to form a gauge-invariant source,

$$\rho = \bar{q}^a(x) \left[P e^{i\int_x^{x+r} A_\mu dx^\mu} \right]_{ab} q^b(x+r) \tag{4.35}$$

which is no longer a matrix, since we have traced over the color indices. But this trace is not actually present in the equation of motion (4.27), which has the explicit form

$$\begin{aligned} \mathbb{D}\cdot\mathbb{E} &= \mathbb{\nabla}\cdot\mathbb{E} - i[\mathbb{A}\cdot,\mathbb{E}] \\ &= \mathbb{\nabla}\cdot\dot{\mathbb{A}} - i[\mathbb{A}\cdot,\dot{\mathbb{A}}] = g^2\rho \end{aligned} \tag{4.36}$$

in $A_0 = 0$ gauge. In this form it is clear that $\mathrm{tr}\,\mathbb{D}\cdot\mathbb{E} = 0$ since every term is proportional to an SU(3) generator. Therefore (as we already knew) only the traceless part of ρ can act as a source for the gluons.

Let's rewrite (4.27) in the SU(3) vector notation I introduced previously,

$$\nabla_i \dot{\vec{A}}_i - iA_i^a \dot{A}_i^b [T^a, T^b] = g^2 \vec{\rho}. \tag{4.37}$$

Using $[T^a, T^b] = if_{abc}T^c$ and rescaling $A \to g^2 A$, this becomes

$$\nabla_i \dot{\vec{A}}_i + g^2 \vec{A}_i{}^\times \dot{\vec{A}}_i = \vec{\rho}. \tag{4.38}$$

Now let $\vec{A}_i = t\,\vec{E}_i$; then since $\vec{E}_i{}^\times \vec{E}_i = 0$, we get Gauss's law $\nabla_i \vec{E}_i = \vec{\rho}$. So it looks like we have succeeded in finding a class of solutions, that looks like just eight copies of the Abelian problem. Not so fast! In electrodynamics, there is no difficulty in setting the magnetic field to zero for a static charge configuration. But in QCD the color field sources itself, and now it is no longer obvious that we can set $B = 0$. Since B contains the term $[A, A]$, it would vanish for special charge distributions where ρ^3 and ρ^8 (whose generators are diagonal) are the only nonzero components. But such solutions are not helpful for understanding the distinctive properties of QCD, in particular the confining potential.

More generally there could be an integration constant, $\vec{A}_i = \vec{a}_i(\vec{x}) + t\,\vec{E}_i$, giving the extra term

$$\nabla_i \vec{E}_i + g^2 \vec{a}_i{}^\times \vec{E}_i = \vec{\rho} \tag{4.39}$$

in Gauss's law. What is the physical significance of $\vec{a}_i$? Recall that

$$B_z = E_{xy} = \partial_x A_y - \partial_y A_x - [A_x, A_y] \tag{4.40}$$

so that $\vec{a}_i(x)$ gives a time-independent contribution to the chromomagnetic field,

$$\begin{aligned} B_z &= \partial_x a_y - \partial_y a_x + t[\partial_x E_y - \partial_y E_x] \\ &- [a_x + tE_x, a_y + tE_y] \\ &= \partial_x a_y - \partial_y a_x - [a_x, a_y] \\ &+ t\left(\partial_x E_y - \partial_y E_x - [E_x, a_y] - [a_x, E_y]\right) - t^2[E_x, E_y] \end{aligned} \tag{4.41}$$

The time-dependent terms still vanish for the special charge distributions ρ^3, ρ^8, while the time-independent one vanishes if in addition a_i^3 and a_i^8 are curl-free.

In electrodynamics, a static electric and magnetic field in temporal gauge are described by $A_i = a_i + tE_i$ with $\nabla \times E = 0$ and $\partial_t a_i = \partial_t E_i = 0$. We can't seem to do that here:

$$\begin{aligned} E_i &= -\partial_i A_0 + \partial_0 A_i + [A_i, A_0] \\ &= \partial_0 A_i \ \text{ in temporal gauge;} \\ B_x &= \partial_y A_z - \partial_z A_y - [A_y, A_z] \end{aligned} \tag{4.42}$$

because the commutator in B generically gives rise to time dependence. This seems to imply that we cannot impose $A_0 = 0$ gauge when charges are present. In electrodynamics it is more common to express a static solution as $\vec{E} = \vec{\nabla} A_0$ in Coulomb gauge where; $\vec{\nabla}\cdot\vec{A} = 0$. Then

$$\vec{\nabla}\cdot\vec{E} \sim \vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \partial_i \frac{x_i}{r^3} = \frac{3}{r^3} - \left(\frac{3}{2}\right)\frac{x_i\,2x_i}{r^5} = 0 \tag{4.43}$$

But we previously showed that it is always possible to go to temporal gauge; why should it matter what gauge we choose?

One reason it could matter is the gauge-covariance of the source. Suppose that ρ was initially static in a gauge where $A_0 \neq 0$. When we transform to temporal gauge, it is no longer static! Instead

$$\Lambda^\dagger \rho \Lambda = \left(P\, e^{i\int^t dt' A_0(t')}\right)^\dagger \rho \left(P\, e^{i\int^t dt' A_0(t')}\right) \tag{4.44}$$

which is t-dependent, unlike in the Abelian case. One might try to fix the problem by rewriting Gauss's law in terms of gauge invariant quantities on the left-hand side of the equation. Using $\mathbb{D}\cdot\mathbb{E} = \mathbb{\nabla}\cdot\mathbb{E} - i[\mathbb{A},\mathbb{E}]$, it would read

$$\mathbb{\nabla}\cdot\mathbb{E} = g^2\rho + i[\mathbb{A},\mathbb{E}] \tag{4.45}$$

But that doesn't work, since $\mathbb{E}$ itself is not gauge invariant!

Eq. (4.35) suggests that it might be possible to find a solution where the charge remains static if we work instead in an axial gauge with $\mathbb{n}\cdot\mathbb{A} = 0$ for some spacelike vector $\mathbb{n}$, for example $A_z = 0$, in the case where the quark and antiquark are separated along the z direction. Then the gauge transformation needed to transform from $A_z \neq 0$ to $A'_z = 0$ is

$$\Lambda = P e^{i\int^z dz' A_z(z')} \tag{4.46}$$

If the initial gauge field was static, $\dot{A}_\mu = 0$, then $\Lambda^\dagger \rho \Lambda$ remains static. In fact, the same argument would have worked in temporal gauge since then $Pe^{i\int^t dt' A_0(t')} = Pe^{itA_0} = e^{itA_0}$, which would be consistent, but then $\vec{E} = 0$.

In summary, it seems to be difficult to find the classical gauge configurations that would explain the origin of the quark-antiquark potential.[14]

5. QCD CONVENTIONS (10-29-87)

In the previous lectures we may have been a bit careless with numerical factors and signs. Let's now try to get all of these right and establish a consistent set of conventions. First, we can verify that the quark Lagrangian should read $\bar{\psi}(i\partial\!\!\!/ - A\!\!\!/)\psi$ to be invariant under the gauge transformations

$$\begin{aligned} \psi &\to \Lambda\psi, \qquad \bar{\psi} \to \bar{\psi}\Lambda^\dagger, \\ A_\mu &\to \Lambda\, A_\mu\, \Lambda^\dagger + i(\partial_\mu \Lambda)\, \Lambda^\dagger \end{aligned} \tag{5.1}$$

Second, we carry out the gauge transformations

$$\begin{aligned} \partial_\mu A_\nu - \partial_\nu A_\mu &\to \Lambda(\partial_\mu A_\nu - \partial_\nu A_\mu)\Lambda^\dagger \\ &+ (\partial_\mu\Lambda)A_\nu\Lambda^\dagger - (\partial_\nu\Lambda)A_\mu\Lambda^\dagger \\ &+ \Lambda A_\nu \partial_\mu \Lambda^\dagger - \Lambda A_\mu \partial_\nu \Lambda^\dagger \\ &+ i\partial_\mu((\partial_\nu\Lambda)\Lambda^\dagger) - i\partial_\nu((\partial_\mu\Lambda)\Lambda^\dagger) \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} [A_\mu, A_\nu] &\to [\Lambda A_\mu \Lambda^\dagger + i(\partial_\mu\Lambda)\Lambda^\dagger, \Lambda A_\nu \Lambda^\dagger + i(\partial_\nu\Lambda)\Lambda^\dagger] \\ &= \Lambda[A_\mu, A_\nu]\Lambda^\dagger + i[\Lambda A_\mu \Lambda^\dagger, (\partial_\nu\Lambda)\Lambda^\dagger] \\ &+ i[(\partial_\mu\Lambda)\Lambda^\dagger, \Lambda A_\nu \Lambda^\dagger] - [(\partial_\mu\Lambda)\Lambda^\dagger, (\partial_\nu\Lambda)\Lambda^\dagger] \end{aligned} \tag{5.3}$$

Using $(\partial_\nu\Lambda)\Lambda^\dagger = -\Lambda\partial_\nu\Lambda^\dagger$, we can show that the terms involving derivatives of Λ cancel in the linear combination

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \tag{5.4}$$

which is therefore the covariant field strength. The chromoelectric and magnetic fields are

$$\begin{aligned} E_i &= F_{0i} = \partial_0 A_i - \partial_i A_0 + i[A_0, A_i] \\ B_x &= F_{yz} = \partial_y A_z - \partial_z A_y + i[A_y, A_z] \end{aligned} \tag{5.5}$$

The full Lagrangian is

$$\mathcal{L} = \frac{1}{2g^2}\mathrm{tr}\, F_{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - A\!\!\!/ - m)\psi \tag{5.6}$$

[14] This statement was not in my notes; it conveys my impression that RPF was explaining from memory the sequence of difficulties he encountered when looking for classical solutions, some time prior to the course. There is no record of these attempts in his personal notes.

Let's vary it with respect to A to find the equation of motion. The variation of the first term is

$$\begin{aligned} \delta\mathcal{L} &= \frac{1}{g^2}\mathrm{tr}\, F_{\mu\nu}\, \delta F_{\mu\nu} + \ldots \qquad (5.7) \\ &= \frac{1}{g^2}\mathrm{tr}\Big(2\delta A_\mu \partial_\nu F_{\mu\nu} + i([\delta A_\mu, A_\nu] + [A_\mu, \delta A_\nu])F_{\mu\nu}\Big) \\ &= \frac{2}{g^2}\mathrm{tr}\Big(\delta A_\mu \partial_\nu F_{\mu\nu} + i[\delta A_\mu, A_\nu]F_{\mu\nu}\Big) \qquad (5.8) \end{aligned}$$

Then[15] writing $A_\mu = A^a_\mu T_a = A^a_\mu(\frac{1}{2}\lambda_a)$,

$$\begin{aligned} g^2\frac{\delta\mathcal{L}}{\delta A^a_\mu} &= 2\mathrm{tr}\Big(T_a\, \partial_\nu F_{\mu\nu} + i[T_a, A_\nu]F_{\mu\nu}\Big) \\ &= \partial_\nu F^a_{\mu\nu} + 2i\, A^b_\nu\, F^c_{\mu\nu}\mathrm{tr}\Big([T^a, T^b]\, T^c\Big) \\ &= \partial_\nu F^a_{\mu\nu} + i[A_\nu, F_{\mu\nu}]^a \end{aligned} \tag{5.9}$$

where we used

$$\mathrm{tr}([T^a, T^b]\, T^c) = \frac{i}{2}f_{abc} = \mathrm{tr}([T^b, T^c]\, T^a) \tag{5.10}$$

Therefore

$$\partial_\nu F^a_{\mu\nu} + i[A_\nu, F_{\mu\nu}]^a = g^2\bar{\psi}T^a\gamma_\mu\psi \tag{5.11}$$

Exercise. Show that in Coulomb gauge $\partial_i A_i = 0$, the Gauss's law constraint becomes[16]

$$\nabla^2 A_0 + 2ig[A_i, \partial_i A_0] - g^2\,[A_i, [A_i, A_0]] = g\rho \tag{5.12}$$

after rescaling $A_\mu \to gA_\mu$. ρ is the matrix charge defined in eq. (4.28).

6. GEOMETRY OF COLOR SPACE* (11-3,5-87)

[17] Our discussion of the QCD Lagrangian has been of a largely algebraic nature to this point, but much intuition can be gained by considering the local color symmetry in geometric terms. At each point in spacetime we imagine there exists a set of axes in the color space,

[15] At this point RPF writes $\delta\mathcal{L}/\delta A^a_\mu$ on the left side, but on the right side gives the variation of $\mathcal{L}$ with respect to $A^{a\bar{b}}_\mu$ labeled by the $(3, \bar{3})$ indices, rather than varying with respect to A^a_μ labeled by the adjoint index a. This gives a result twice as large as it should be (due to the normalization of the generators), which RPF recognizes as being wrong and therefore concludes that the gluon kinetic term should really be normalized as $(1/4g^2)\mathrm{tr}\, F^2$. This may be another case of him extemporizing. I have corrected the derivation here.

[16] In the lectures, RPF derives this but I leave it as an exercise. Part of the derivation involves assuming that $\partial_0 A_i = 0$ in the commutator $[A_i, \partial_0 A_i]$, which seems not generally true.

[17] This section, which was revised by RPF, combines lectures 6 and 7, given on Nov. 3 and 5, 1987. It repeats some material that was presented earlier. I retained the redundancies in the interest of historical accuracy.

which may vary in its relative orientation from place to place. This freedom to rotate color frames independently at each point is embodied in the SU(3) transformation matrices $\Lambda(x)$, under which a quark transforms as

$$\psi'(x) = \Lambda(x)\psi(x)\,. \tag{6.1}$$

Since one rotation may be followed by another,

$$\begin{aligned}\psi''(x) &= \Lambda'(x)\psi'(x) = \Lambda'(x)\Lambda(x)\psi(x)\\ &\equiv \Lambda''(x)\psi(x)\,;\end{aligned}$$

they form a group, with $\Lambda'' = \Lambda'\Lambda$ being the group multiplication law. Requiring that Λ not change the length of a color vector is equivalent to demanding that $\Lambda^\dagger\Lambda = 1$. Thus the Λ's would represent the group U(3) of unitary 3×3 matrices. However U(3) contains a U(1) subgroup, matrices of the form $e^{i\theta}\mathbb{1}$, which would give rise to an additional long-range interaction like the electromagnetic force. To eliminate this we note that

$$\det\Lambda'' = \det\Lambda'\det\Lambda$$

represents the U(1) transformations (it is Abelian), so we should make the restriction

$$\det\Lambda = 1,$$

i.e., Λ is a *special* unitary matrix, hence the group is SU(3).

The transformation law for the gluon field has a less immediately obvious interpretation than that for the quarks, eq. (6.1). For infinitesimal rotations $\Lambda = 1+i\mathbf{a}$,[18]

$$\begin{aligned}A'_\mu &= (1-i\mathbf{a})A_\mu(1+i\mathbf{a}) + (1-i\mathbf{a})\partial_\mu(1+i\mathbf{a})\\ &= A_\mu - i[\mathbf{a},A_\mu] + i\partial_\mu\mathbf{a}\\ &= A_\mu + iD_\mu\mathbf{a}\end{aligned} \tag{6.2}$$

where D_μ is the covariant derivative. How can this be understood geometrically?

To answer this, it must first be realized that there is, *a priori*, no way of telling whether a color frame at point x is parallel to one at $x+\Delta x$, because the color space is completely unrelated to spacetime. An analogy is trying to choose local tangent frames on a curved space, such as the surface of a two-sphere, that are "parallel" to each other. It is not possible to do without defining a law of parallel transport for vectors, so that we know what it means for two vectors at different locations to be parallel. Similarly in QCD one needs a rule for comparing orientations of nearby color frames. This is the function of the gauge field $A_\mu(x)$, in much the same way as the metric tensor (to be more precise, the Christoffel symbol) defines parallel transport in the geometry of curved space.

[18] Neither RPF nor I noticed the inconsistency with eq. (5.1), which is the correct version having $i\partial_\mu$ instead of ∂_μ in the first line.

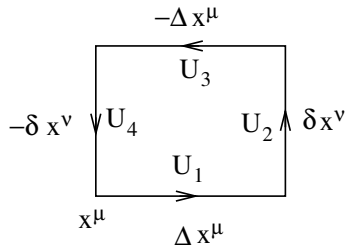


FIG. 13: Parallel transport of a quark around a closed loop.

Define the relative orientation between two nearby color frames, at x and $x+\Delta x$, to be given by the rotation matrix

$$U(x,\Delta x) = 1 + iA_\mu(x)\Delta x^\mu\,. \tag{6.3}$$

Now suppose that every set of axes is rotated by $\Lambda(x)$, depending on the position x. Then the new transformation relating the frames at x and $x+\Delta x$ is

$$\begin{aligned}U'(x,\Delta x) &= \Lambda^\dagger(x+\Delta x)U(x,\Delta x)\Lambda(x)\\ &= 1 + iA'_\mu(x)\Delta x^\mu\end{aligned} \tag{6.4}$$

It follows that

$$A'_\mu(x) = \Lambda^\dagger(x)A_\mu(x)\Lambda(x) + i\Lambda^\dagger(x)\partial_\mu\Lambda(x) \tag{6.5}$$

which shows that our geometric interpretation of $A_\mu(x)$ agrees with its previously determined transformation law.

If one was to take a quark at x with color vector $\vec{q}$ and parallel-transport it to $x+\Delta x$, its color would change to $U(x,\Delta x)\vec{q}$. Of particular interest is the change in $\vec{q}$ when transported around a closed loop, such as the one shown in fig. 13. Let $U_1 = U(x,\Delta x)$, $U_2 = U(x+\Delta x,\delta x)$, $U_3 = U(x+\Delta x+\delta x,-\Delta x)$, $U_4 = U(x+\delta x,-\delta x)$. The transformation of $\vec{q}$ in going around the loop is

$$\vec{q}\,' = U_4U_3U_2U_1\vec{q} = U_{\rm tot}\vec{q} \tag{6.6}$$

One notices that $U_{\rm tot}(x)$ has a simple transformation under local SU(3) rotations,

$$U_{\rm tot}(x) \to \Lambda^\dagger(x)U_{\rm tot}(x)\Lambda(x)\,,$$

which is just how the field strength $F_{\mu\nu}(x)$ transforms. This is not an accident: if you expand $U_{\rm tot}$ in terms of the gauge field as in (6.3), you will find that

$$U_{\rm tot} = 1 + iF_{\mu\nu}(x)\Delta x^\mu\delta x^\nu \tag{6.7}$$

plus terms of order $(\Delta x)^2$ and $(\delta x)^2$. Notice that $\Delta x^\mu\delta x^\nu$ is the area of the loop. So $F_{\mu\nu}(x)$ tells us how much color rotation a quark suffers under transformations around infinitesimal loops. It is analogous to the Riemann tensor, which does the same thing for vectors in curved space.

With the concept of parallel transport in hand, covariant differentiation becomes quite transparent. If $\psi(x)$ is a quark field, it is not $\psi(x+\Delta x) - \psi(x)$ that is of

physical interest, because this includes the difference due to arbitrary orientations of the local color axes. We should rather compare $\psi(x+\Delta x)$ with $\psi(x)$ transported to $x+\Delta x$. Therefore define the covariant derivative as

$$\Delta x^\mu D_\mu \psi(x) = \psi(x+\Delta x) - U(x,\Delta x)\psi(x)$$

or equivalently

$$D_\mu \psi(x) = (\partial_\mu - iA_\mu)\psi(x)\,. \tag{6.8}$$

Similarly for the field strength,

$$\Delta x^\alpha D_\alpha F_{\mu\nu}(x) = F_{\mu\nu}(x+\Delta x) - U^\dagger(x,\Delta x)F_{\mu\nu}(x)U(x,\Delta x)$$

which implies

$$D_\alpha F_{\mu\nu}(x) = \partial_\alpha F_{\mu\nu}(x) - i[A_\alpha, F_{\mu\nu}] \tag{6.9}$$

Even as seemingly abstract an equation as the Bianchi identity can be understood geometrically. This is one of the statements you were asked to prove previously,

$$D_\alpha F_{\beta\gamma} + D_\gamma F_{\alpha\beta} + D_\beta F_{\gamma\alpha} = 0\,. \tag{6.10}$$

For concreteness, let $(\alpha,\beta,\gamma) = (x,y,z)$. Then the first term is $D_x F_{yz}$. In terms of fig. 14, this is the change in color axis orientation around the top loop minus that of the bottom loop. The use of D_z rather than ∂_z means that the bottom loop was parallel transported to the position of the top loop before making the subtraction. The contribution to $D_z F_{xy}$ from each link of the cube is denoted by a line with an arrow that shows the relative sign of the contribution. From the figure, it is easy to see that when the remaining terms in (6.10) are included, each link will contribute twice, once in each direction. Therefore the sum is zero.

The matrix $U(x,\Delta x)$ that connects nearby color axes can be used to construct the rotation connecting color frames that are separated by a finite distance. Choose a path connecting the two points and divide it into small increments, labeled by x_i^μ, such that $\Delta x_i^\mu = x_{i+1}^\mu - x_i^\mu$ as shown in fig. 15. Then the rotation matrix between x_0^μ and x_f^μ is

$$S(x_0,x_f) = \prod_i \left(1 + iA_\mu(x_i)\Delta x_i^\mu\right)\,. \tag{6.11}$$

As $\Delta x_i^\mu \to 0$, this becomes equivalent to $\prod_i e^{iA_\mu(x_i)\Delta x_i^\mu}$. We would like to write it as $\exp(i\int A_\mu\, dx^\mu)$, which would

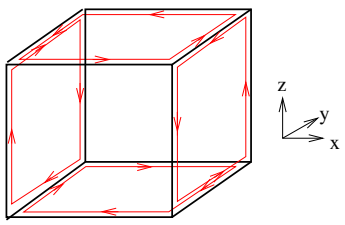


FIG. 14: Geometric interpretation of the Bianchi identity.

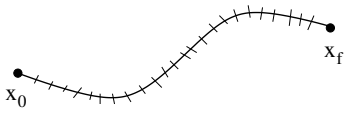


FIG. 15: Path connecting two points in spacetime.

be true if the A_μ were numbers, but since the A_μ don't commute at different positions, we cannot add the exponents. Instead one defines the path ordering operator

$$P e^{b_2+b_1} = e^{b_2} e^{b_1}\,,$$

where it is understood that b_2 is farther along the path than b_1. Therefore

$$\begin{aligned} S(x_0,x_f) &= P\exp\left(i\int_{x_0}^{x_f} dx^\mu\, A_\mu\right) \\ &= 1 + i\int_{x_0}^{x_f} dx^\mu\, A_\mu \\ &\quad - \int_{x_0}^{x_f} dx^\mu \int_x^{x_f} dx'^\nu\, A_\nu(x')A_\mu(x) + \cdots \end{aligned} \tag{6.12}$$

Notice that S is by no means unique; it depends upon the path chosen. Under a gauge transformation however,

$$S(x_0,x_f) = \Lambda^\dagger(x_f) S \Lambda(x_0) \tag{6.13}$$

regardless of the path.

Exercise. Using the definition (6.12), show that

$$P\exp\left(-\int_{x_0}^{x_f} dx^\mu\, \Lambda^\dagger(x)\partial_\mu\Lambda(x)\right) = \Lambda^\dagger(x_f)\Lambda(x_0)$$

Hint: consider the differential equation satisfied by (6.12) for $\partial S/\partial x^\mu$.

The connection S is useful for making bilocal operators gauge invariant. For example, in the full interacting theory, the two-point function for the field strength vanishes because of gauge invariance, since

$$\begin{aligned} &\langle 0|F_{\alpha\beta}(x_0)F_{\mu\nu}(x_f)|0\rangle = \\ &\quad \langle 0|\Lambda^\dagger(x_0)F_{\alpha\beta}(x_0)\Lambda(x_0)\Lambda^\dagger(x_f)F_{\mu\nu}(x_f)\Lambda(x_f)|0\rangle \end{aligned}$$

for an arbitrary $\Lambda(x)$. This can only be satisfied if $\langle FF\rangle = 0$. However the function

$$\mathrm{tr}\left(S^\dagger(x_0,x_f)F_{\alpha\beta}(x_0)S(x_0,x_f)F_{\mu\nu}(x_f)\right)$$

is gauge invariant, and has a meaningful nonvanishing expectation value.

The statement that $\langle FF\rangle = 0$ for F's at distinct points implies some way of defining expectation values without tampering with the gauge symmetry of the path integral over A_μ, to be discussed later on. In practice, it is necessary to choose a condition that fixes the gauge, by associating $F_{\mu\nu}(x)$ with a unique vector potential $A_\mu(x)$. For example, it is always possible to demand that $A_0 = 0$ by transforming $A_\mu \to \Lambda^\dagger A_\mu \Lambda + i\Lambda^\dagger\partial_\mu\Lambda$, where

$$\Lambda^\dagger A_0 \Lambda + i\Lambda^\dagger\partial_\mu\Lambda = 0$$

One can show that the solution to this equation is

$$\Lambda = T\,\exp\left(i\int^t A_0(t')\right)dt'$$

where T is the same as P but for a purely timelike path.

Another gauge condition that is conceptually useful is to minimize the quantity

$$\int d^4x\,\mathrm{tr}(A_\mu(x))^2 \tag{6.14}$$

In this gauge, $P\,\exp(i\int dx\,A)$ does not change much if the path is varied only slightly. Therefore it is possible to define a global orientation for color axes on short enough distance scales: a quark that looks red at point x will still look red after parallel transport if it is not carried too far. Because of this it makes some sense to say that quarks of the same color repel each other, whereas quarks that are antisymmetric in their colors attract each other, as will be shown in the next lecture. In an arbitrary gauge it would not be meaningful to say that two blue quarks repel each other, unless they were at the same position, since what is blue at one place may not be blue at another.

The above choice of gauge is closely related to another more familiar one. Under an infinitesimal gauge transformation $A_\mu \to A_\mu + D_\mu\alpha$, the change in (6.14) is

$$2\int d^4x\,\mathrm{tr}A_\mu D_\mu\alpha = -2\int d^4x\,\mathrm{tr}(\alpha\,D_\mu A_\mu) \tag{6.15}$$

If $\int \mathrm{tr}(A_\mu^2)$ is at a minimum, then (6.15) must vanish for all $\alpha(x)$. This implies

$$D_\mu A_\mu = \partial_\mu A_\mu = 0 \tag{6.16}$$

However the two gauges are not equivalent, because the first one asks for the absolute minimum of $\int \mathrm{tr}(A_\mu^2)$, whereas $\partial_\mu A_\mu = 0$ only requires that $\int \mathrm{tr}(A_\mu)^2$ be at a local minimum. Therefore $\partial_\mu A_\mu = 0$ may have many solutions, and it does not uniquely fix the gauge. This problem was first discussed by Gribov in the context of the path integral.

6.1. Omitted material

[19] A synopsis of the popular gauge choices is

$$\begin{aligned} A_t &= 0 \quad \text{Weyl} \\ \partial_\mu A_\mu &= 0 \quad \text{Lorentz} \\ \mathbb{\nabla}\cdot\mathbb{A} &= 0 \quad \text{Coulomb} \end{aligned} \tag{6.17}$$

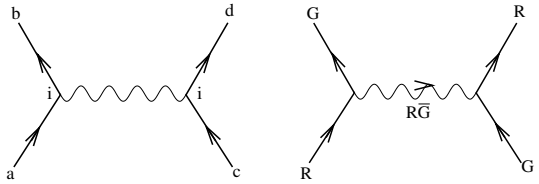


FIG. 16: (a) Left: quark-quark scattering by gluon exchange. (b) Right: same, with particular choice of colors.

In addition, there is an analog to (6.14) due to Mandula, which is to minimize $\int d^3x\,\mathbb{A}:\mathbb{A}$.

Exercise. Show that $E_{\mu\nu}\cdot\widetilde{E}_{\mu\nu} \equiv \epsilon_{\mu\nu\alpha\beta}E_{\mu\nu}\cdot E_{\alpha\beta}$ could be added to the Lagrangian (usually written as the action $\theta\int d^4x\,E_{\mu\nu}\widetilde{E}_{\mu\nu}$), but it makes no contribution to the equations of motion: it is a total derivative.

7. SEMICLASSICAL QCD* (11-10-87)

I know you are eager to move on to the *quantum* theory of chromodynamics, now that we have studied it at the classical level, but there always has to be some professor deterring you by saying "before we do that, let's look at such-and-such!" Accordingly, before we quantize QCD I want to discuss a somewhat tangential but very important issue: can we explain the properties of the hadrons, even *qualitatively*, with the theory of QCD? That is, we would like to see that we are at least going in roughly the right direction before we invest all our effort in it. For example, it would be quite discouraging if at the lowest level of analysis QCD predicted that the three quarks in a baryon will want to fly apart.

But we shall see that it *does* work, and we won't even have to do that much work ourselves to see it, if we just remember a few things from quantum *electro*dynamics. This is because at lowest order in the coupling constant g, the interaction between two quarks is given by essentially the same Feynman diagram as that for electron-electron scattering, fig. 16. The only difference is that in the case of QCD, each vertex comes with a group theory factor λ^i_{ab} and λ^i_{cd} to account for the fact that the quarks are changing color when they exchange a gluon of color i. The sum over intermediate gluon colors then gives a factor $\vec{\lambda}_{ab}\cdot\vec{\lambda}_{cd}$ in the amplitude.[20]

Of course we know that fig. 16(a) is not a good approximation for the quark-quark scattering in a hadron, because the coupling is large. But we just want to see that it is going in the right *direction*, when we do make this approximation. Knowing that a single photon exchange gives rise to the Coulomb potential in electrodynamics, we can immediately write the quark-quark potential from

[19] This material appears in my original notes but was omitted from the revised version above.

[20] These should be accompanied by extra factors of $1/2$ from $T^a = \lambda^a/2$.

fig. 16(a) as

$$V(r) = \frac{g^2}{r}\,\vec{\lambda}_{ab}\cdot\vec{\lambda}_{cd} \tag{7.1}$$

Now all that remains is to evaluate the $\vec{\lambda}\cdot\vec{\lambda}$ factor in the color channels appropriate for baryons and mesons. This could be done by using fancy group theory techniques, but I find it useful to take a more simple-minded approach at first. It is not necessary to know group theory—all one needs to know is that there are three colors!

Let us suppose that quark 1 (on the left) starts out being red, and converts to green by emitting a red-antigreen gluon. In order to conserve color, quark 2 must have started out being green and turn to red when it absorbs the gluon. Using the convention

$$(\mathrm{R},\ \mathrm{G}\ \ \mathrm{B})^T = (\text{red, green, blue})^T \tag{7.2}$$

for the components of a color vector, there are two λ matrices contributing to the process shown in fig. 16(b), namely λ^1 and λ^2,

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.3}$$

due to the fact that a $R\bar{G}$ gluon corresponds to a particular linear combination, $\lambda^1 + i\lambda^2$. Therefore the contribution to $\vec{\lambda}_{ab}\cdot\vec{\lambda}_{cd}$ from fig. 16(b) is

$$1\cdot 1 + i(-i) = 2 \tag{7.4}$$

Now suppose that the colors of the initial state quark did not change. This could happen if gluons corresponding to the color-diagonal generators λ^3 and λ^8 were exchanged,

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad \lambda_8 = \sqrt{\tfrac{1}{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{7.5}$$

For example, the contribution to $\vec{\lambda}\cdot\vec{\lambda}$ from fig. 17 is

$$1\cdot 0 + \frac{1}{\sqrt{3}}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{2}{3} \tag{7.6}$$

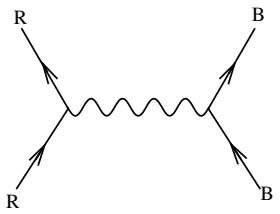


FIG. 17: A color-conserving q-q scattering process.

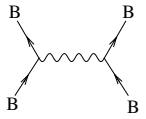
From (7.4) and (7.6) we could guess that the general expression for $\vec{\lambda}\cdot\vec{\lambda}$ is

$$\vec{\lambda}_{ab}\cdot\vec{\lambda}_{cd} = 2\,P^{c.e.}_{ab,cd} - \frac{2}{3}\delta_{ab}\delta_{cd} \tag{7.7}$$

where $P^{c.e.}$ is the color-exchange operator,

$$P^{c.e.}_{ab,cd} = \begin{cases} 1 & \text{if } a=d \text{ and } b=c \\ 0 & \text{otherwise} \end{cases} \tag{7.8}$$

As a check, look at the graph



$$: \left(-\frac{2}{\sqrt{3}}\right)\left(-\frac{2}{\sqrt{3}}\right) = \frac{4}{3} \tag{7.9}$$

which has only a λ^8-type gluon. This agrees with (7.7) since $2P^{c.e.}_{B\bar{B},B\bar{B}} - \frac{2}{3}\delta_{B\bar{B}}\delta_{B\bar{B}} = 4/3$.

So far we have not concerned ourselves about whether the initial state was symmetric or antisymmetric in color. If it was antisymmetric, it would be an eigenstate of the color exchange operator with eigenvalue -1. (Of course, it would also be an eigenstate of the identity operator $\delta_{ab}\delta_{cd}$ with eigenvalue $+1$.) On such a state,

$$\vec{\lambda}\cdot\vec{\lambda}\,|\psi\rangle = \left(2P^{c.e.} - \tfrac{2}{3}\right)|\psi\rangle = -\tfrac{8}{3}|\psi\rangle\,. \tag{7.10}$$

It means that in the antisymmetric channel, the quark-quark potential is

$$V(r) = -\frac{8}{3}\,\frac{g^2}{r}\,. \tag{7.11}$$

The sign is important: since electron-electron scattering is repulsive, the relative sign here tells us that quarks that are antisymmetric in their colors *attract*. This is precisely what we want: in a baryon the quarks are in a completely antisymmetric state, which is the only way to make a color-neutral object out of three color triplets. So we understand, in a rough way, why protons, neutrons, Δ's, *etc.* exist.

The mesons can be understood similarly. In this case the initial state is color symmetric,

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(R\bar{R} + B\bar{B} + G\bar{G}\right). \tag{7.12}$$

If we focus on the $R\bar{R}$ part, there are three graphs to consider, fig. 18. They contribute $-\vec{\lambda}\cdot\vec{\lambda}$ factors (the extra minus sign coming from the coupling of vectors to

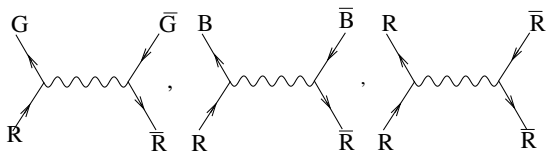


FIG. 18: Scattering of q-$\bar{q}$ within a meson.

antiparticles) of

$$-(1\cdot 1 + i(-i)) = -2, \quad -2, \quad -\left(1\cdot 1 + \tfrac{1}{\sqrt{3}}\cdot\tfrac{1}{\sqrt{3}}\right) = -\tfrac{4}{3} \tag{7.13}$$

respectively. Obviously the $B\bar{B}$ and $G\bar{G}$ parts of $|\psi\rangle$ do the analogous thing, so that the quark-antiquark potential for color-symmetric states is

$$V(r) = -\frac{16}{3}\frac{g^2}{r}\,. \tag{7.14}$$

Graphs like fig. 18 give an *attractive* potential for electrons and positrons in QED, and we see that quarks and antiquarks in a meson must also attract each other.

One notices that the $q\bar{q}$ force in mesons is twice as strong as the qq force in baryons. But it is interesting to note that the total force *per quark* is the same in each system, since

$$\frac{1}{2 \text{ quarks}}\left(\frac{16}{3}\right) = \frac{8}{3} \text{ for mesons} \tag{7.15}$$

and

$$\frac{1}{3 \text{ quarks}} \times (3 \text{ pairwise forces }) \times \left(\frac{8}{3}\right) = \frac{8}{3} \text{ for baryons} \tag{7.16}$$

Problem. Show that the general $q\bar{q}$ interaction due to one-gluon exchange (summed over gluon colors) can be written as

$$\tfrac{2}{3}\,\mathbb{1} - 6|s\rangle\langle s|$$

where $\mathbb{1}$ is the identity operator in color space, and $|s\rangle\langle s|$ projects onto the color singlet state,

$$|s\rangle = \tfrac{1}{\sqrt{3}}\left(R\bar{R} + B\bar{B} + G\bar{G}\right)$$

7.1. Spin-spin interactions

So far, so good: QCD explains why the hadrons exist, even at this crude level of approximation where g was taken to be small. Now we would like to see if it explains some more detailed observations, like the nondegeneracy of the Δ^0 and the neutron:

$$\begin{aligned} \Delta^0 &: \text{—} \quad 1232\,\text{MeV};\ |\psi\rangle = (udd)(\uparrow\uparrow\uparrow) \\ N &: \text{—} \quad 935\,\text{MeV};\ |\psi\rangle = (udd)\tfrac{1}{\sqrt{6}}(2\downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow) \end{aligned} \tag{7.17}$$

If the masses were coming solely from the constituent masses of the quarks, these states would be degenerate due to their identical quark content. The only difference between them seems to be their spin wave functions. Therefore their mass splitting must be due to spin-dependent forces. This comes as no surprise since the photon-exchange graph gives a spin-spin interaction as well as the Coulomb interaction in QED. Let us recall what the sign of the force is in electromagnetism. Specifically, if we could make an s-wave from two electrons, would their spins tend to align or anti-align? We draw the second spin in the magnetic field of the first, with both of them pointing up; see fig. 19. When the second spin is beside the first, the interaction energy is positive, since spin 2 would prefer to flip so as to align with the $\vec{B}$ field. When spin 2 is above spin 1, the interaction energy is negative. To make an s-wave we must average over the positions of spin 2 relative to spin 1 in a spherically symmetric way. One can show that at any nonzero radius, the interaction energy is zero when this is done.

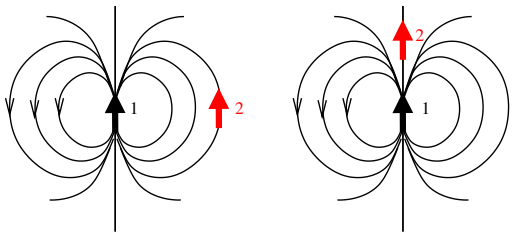


FIG. 19: An electron spin (2) in the dipole field produced by another electron (1).

However, zero is not the correct answer, as we already know. The problem is that the spins have been idealized as pointlike objects. If they actually have a small but finite spatial extent, the magnetic field of spin 1 will look like fig. 20. Because of the interior region, the integration over positions of spin 2 give a net positive interaction energy, and two electrons in an s-wave tend to align. This is known as the Fermi interaction; it is the first term in the expression

$$\begin{aligned} V_{\text{spin}}(\vec{r}) \;\propto\; & -\frac{8\pi}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2\,\delta^{(3)}(\vec{r}) \\ & + \frac{1}{r^3}\left(\vec{\sigma}_1\cdot\vec{\sigma}_2 - 3(\vec{\sigma}_1\cdot\hat{r})(\vec{\sigma}_2\cdot\hat{r})\right). \end{aligned} \tag{7.18}$$

The second term comes from the exterior region that we discussed previously; its angular average is zero, as we noted.

From this we can deduce that the color magnetic moment interaction energy for two quarks, in a relative s-state, is *positive* if the spins are aligned and the colors are antisymmetric. This is because the sign of the spin force relative to that of the Coulomb force is determined by the Lorentz indices of the diagram in fig. 16, so this relative sign must be the same for QCD and QED. Hence the Δ is heavier than the N—it takes more energy to line up all the spins.

What about the Σ^0 and the Λ? They have identical quark content, and equal numbers of aligned spins, yet the Σ^0 is more massive. However, the spin wave functions are not the same,

$$\begin{aligned} |\Sigma^0\rangle &= (sud)\,\tfrac{1}{\sqrt{6}}(2\downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow) \\ |\Lambda\rangle &= (sud)\,\tfrac{1}{\sqrt{2}}(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow) \end{aligned} \tag{7.19}$$

We must remember that the magnetic moment of a quark is inversely proportional to its mass, so the s quark has

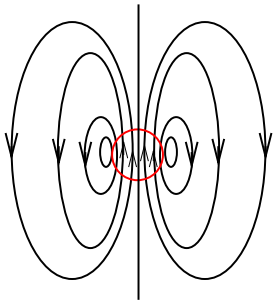


FIG. 20: Effect of a nonpointlike spin 2 (red) overlapping the central region of the magnetic field produced by spin 1.

smaller spin interactions. In the limit that its magnetic moment is neglected, the Σ^0 is 4/6 aligned and 2/6 anti-aligned in its spins, whereas the Λ is completely anti-aligned. Hence the Λ is expected to be lighter, as observed.

To be more quantitative, suppose the spin-spin coupling has strength a_{qq}, a_{qs}, a_{ss} between two light quarks, one light and one strange, and two strange quarks, respectively. Then the spin-spin coupling for Σ^0 and Λ particles is

$$\hat{\mathcal{O}} = a_{qq}\,(\sigma_u{\cdot}\sigma_d) + a_{qs}\,(\sigma_s{\cdot}\sigma_u + \sigma_s{\cdot}\sigma_d) \tag{7.20}$$

where $a_{qs} < a_{qq}$, or equivalently, defining a spin-exchange operator $P^{s.e.}_{qq'}$,

$$\hat{\mathcal{O}} = a_{qq}\,(2P^{s.e.}_{ud} - 1) + a_{qs}\,(2P^{s.e.}_{su} - 1) + a_{qs}\,(2P^{s.e.}_{sd} - 1) \tag{7.21}$$

Acting on Σ^0,

$$\begin{aligned} P^{s.e}_{ud}(2\downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow) &= 2\downarrow\uparrow\uparrow - \uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow, \\ P^{s.e}_{su}(2\downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow) &= 2\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow - \uparrow\uparrow\downarrow, \\ P^{s.e}_{sd}(2\downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow) &= 2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow \end{aligned} \tag{7.22}$$

Therefore

$$\begin{aligned} \hat{\mathcal{O}}|\Sigma^0\rangle &= [(2-1)a_{qq} + (-2-2)a_{qs}]\,|\Sigma^0\rangle \\ &= (a_{qq} - 4\,a_{sq})\,|\Sigma^0\rangle \end{aligned} \tag{7.23}$$

Similarly one finds that

$$\hat{\mathcal{O}}|\Lambda\rangle = [(-2-1)a_{qq} + (2-2)a_{qs}]\,|\Lambda\rangle = -3|\Lambda\rangle \tag{7.24}$$

Therefore the mass difference is $M_\Sigma - M_\Lambda = 4(a_{qq} - a_{qs})$, which is positive since $a_{qq} > a_{qs}$.

Problem. Find the mass splittings of the rest of the baryon $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet states. Assuming that

$$a_{qq} - a_{qs} = a_{qs} - a_{ss}$$

prove the Gell-Mann–Okubo formula,

$$2(M_\Xi + M_N) = 3M_\Lambda + M_\Sigma$$

A similar analysis can be done for the mesons, and it is observed that the level splittings of heavy quarkonium excitations, such as the ψ and Υ systems, are similar to those of positronium. However there is an interesting distinction between the spin forces in quarkonium and those of positronium. In the latter an extra contribution to the spin-spin interaction arises from the annihilation diagram, fig. 21. However, at lowest order in g, no such process can occur for quarkonium. This is because the $q\bar{q}$ pair in a meson forms a color singlet, which cannot annihilate into a colored object like a gluon. It cannot even annihilate into a pair of gluons, for it is spin 1^-, a state not available to two gluons. It requires at least three gluons, which means a high power of the coupling constant, g^6, which is rather small at the scale of the separation between a heavy q and $\bar{q}$ in the ψ or Υ. Also the numerical coefficient of the annihilation amplitude is small, making the width for disintegration of ψ or Υ into hadrons quite narrow. Hence the OZI rule is understood for these particles, at least.

References[21]

A. De Rujula, H. Georgi and S. L. Glashow, "Hadron Masses in a Gauge Theory," Phys. Rev. D **12**, 147 (1975). doi:10.1103/PhysRevD.12.147

T. Appelquist, R. M. Barnett and K. D. Lane, "Charm and Beyond," Ann. Rev. Nucl. Part. Sci. **28**, 387 (1978). doi:10.1146/annurev.ns.28.120178.002131

N. Isgur, "Hadronic structure with QCD: From α to ω (via ψ and Υ)," AIP Conf. Proc. **81**, 1 (1982). doi:10.1063/1.33447

In the last reference, Isgur supposed that the potential between quarks in a hadron is $V(r_{ij}) = \frac{1}{2}kr^2_{ij} + U(r_{ij})$ where U is a perturbation. One needs to evaluate $\langle U\rangle$ and $\langle U|x_i - x_j|^2\rangle$ expectation values. This gives two parameters to fit the spectrum. He discusses the fact that even heavy quark systems have sufficiently large wave

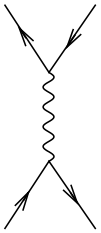


FIG. 21: Annihilation diagram for positronium, not present for quarkonium.

[21] The following references (here modernized to include DOIs) and discussion of Isgur's work are present in my original notes, but somehow got omitted from the revised versions submitted to RPF.

functions at large distances such that the linear term in $V(r) = -\alpha/r + br$ is important. But the whole potential approach is approximate for various reasons, including relativistic effects and annihilation/creation processes.[22]

8. QUANTIZATION OF QCD* (11-12-87)

We now turn to the quantization of QCD. Recall that the action for the gluon fields, interacting with a non-dynamical source, is

$$S[A] = \int \left(\frac{1}{4g^2}\, \vec{F}_{\mu\nu}\cdot\vec{F}_{\mu\nu} + \vec{J}_\mu\cdot\vec{A}_\mu \right) d^4x \tag{8.1}$$

I will assume that you are familiar with the path integral formulation of quantum mechanical amplitudes. If A_i and A_f are the initial and final configurations of the gluon field, the transition amplitude for going from A_i to A_f is

$$K[A_f, A_i] \equiv \int_i^f \mathcal{D}A\, e^{iS[A]} \tag{8.2}$$

where the integral is supposed to be over all field configurations $A^a_\mu(\vec{x}, t)$ such that $A^a_\mu(\vec{x}, t_i) = A^a_{i,\mu}(\vec{x})$ and $A^a_\mu(\vec{x}, t_f) = A^a_{f,\mu}(\vec{x})$. Formally, the measure is defined as an infinite product over all points in spacetime between $t = t_i$ and $t = t_f$,

$$\mathcal{D}A = \prod_{\vec{x},t,\mu} d^8A_\mu(\vec{x}, t) \tag{8.3}$$

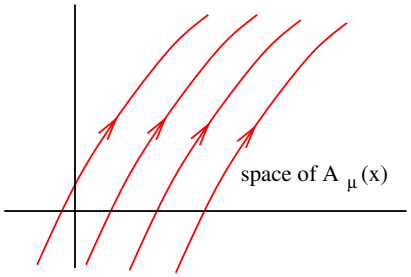
The eight-dimensional measure d^8A_μ simply means the product over the eight components of color, $\prod_i dA^i_\mu$.[23] The action (8.1) is invariant under the local color gauge transformations of A_μ,

$$A'_\mu = \Lambda^\dagger A_\mu \Lambda + i\Lambda^\dagger \partial_\mu \Lambda \tag{8.4}$$

This implies that the measure $\mathcal{D}A$ is also gauge invariant, since it transforms as $dA'_\mu = \Lambda^\dagger dA_\mu \Lambda$ at each point, and the Jacobian of this transformation is trivial when we consider d^8A_μ. **Problem.** Prove this.[24]

Now as you know, this path integral is plagued with infinities. One rather trivial kind is the infinite volume of spacetime. Another sort, the ultraviolet divergences, comes from the uncountably infinite dimensional nature of the measure, an integral for each point of spacetime. This kind I want to ignore for the moment—it can be cured by approximating spacetime as a discrete lattice, in some gauge invariant way. This has been discussed by Wilson, and we shall describe it later. It is the basis for a numerical method to evaluate the path integral.

But in QCD we are still left with another infinity, due to the gauge symmetry itself. If we represent the space of functions $A_\mu(x)$ in two dimensions, we have trajectories of gauge fields that are related to one another by local color rotations:



This means that the expectation value of a physically relevant operator, that is gauge invariant, will diverge like the volume of local gauge transformations in function space,

$$\int \mathcal{D}A\, e^{iS[A]} \sim \int \mathcal{D}\Lambda(\vec{x}, t) = \infty \tag{8.5}$$

Put another way, there are directions in which A can change (*i.e.,* by gauge transformations) for which the integrand is invariant and the region of integration is infinite.

However we can get a finite and meaningful result by defining a kind of expectation value of any gauge invariant functional $F[A]$ of the field, and limiting ourselves to computing only such quantities,

$$\langle F \rangle = \lim_{\mathcal{R}\to\infty} \frac{\int \mathcal{D}A\, F\, e^{iS[A]}}{\int \mathcal{D}A\, e^{iS[A]}} \tag{8.6}$$

where $\mathcal{R}$ is a relatively finite region in the space of gauge fields; for example we might limit the range of each A_μ to be $[-M, +M]$ for some large value of M, the same in the numerator and denominator, and then take the limit as $M \to \infty$. (I say "relatively finite" because there are still infinitely many integration variables, one for each point in spacetime, unless we go to a lattice.) Gauge invariance is broken temporarily by this procedure because $|A'_\mu|$ need not be less than M even if $|A_\mu|$ is, but this should be no problem in the limit $\mathcal{R} \to \infty$.

Having found a finite and gauge-invariant definition of amplitudes, we are free to choose a gauge. A convenient choice is $A_0 = 0$. Recall that it is always possible to reach this gauge from a configuration where $A_0^{\rm old} \neq 0$, using the transformation matrix

$$\Lambda(\vec{x}, t) = P \exp\left(i \int^t A_0^{\rm old}(\vec{x}, t')\right) dt' \,. \tag{8.7}$$

In general, the path integral measure $\mathcal{D}A^{\rm old}_\mu$ changes by a Jacobian factor when we transform the variables

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[22] This paragraph, present in my notes, was omitted from the original revised version.

[23] This sentence may seem extraneous, but RPF was correcting my misconception that the Haar measure for the group manifold was somehow incorporated.

[24] The Jacobian matrix is $\frac{1}{2}\mathrm{tr}(\lambda^a \Lambda^\dagger \lambda^b \Lambda)$. One can show its determinant is trivial by considering an infinitesimal transformation to leading order, and using $\det = \exp \mathrm{tr} \ln$.

$A_\mu^{\rm old} \to A_\mu^{\rm new}$, where $A_0^{\rm new} = 0$. How can this be? The measure was supposed to be gauge invariant! But this is true only for gauge transformations that are *independent of* A_μ. Nevertheless, we will show that for the special case of (8.7), the Jacobian is still almost trivial, even though $\Lambda(\vec{x},t)$ depends on A_μ. This is because Λ depends only on A_0, and thus for the three space directions at least, $d^8A'_m = d^8A_m$ (we use n, m, *etc.* for the spatial components of ν, μ, while the time component is 0.) The expectation value of an operator can now be written as

$$\begin{aligned} &\frac{\int_{\mathcal{R}^{\rm old}} e^{iS[\mathbb{A}^{\rm old},A_0^{\rm old}]} F[\mathbb{A}^{\rm old},A_0^{\rm old}] \prod_{n=1}^3 \mathcal{D}A_n^{\rm old}\, \mathcal{D}A_0^{\rm old}}{\int_{\mathcal{R}^{\rm old}} e^{iS[\mathbb{A}^{\rm old},A_0^{\rm old}]} \prod_{n=1}^3 \mathcal{D}A_n^{\rm old}\, \mathcal{D}A_0^{\rm old}} \\ = \;& \frac{\int_{\mathcal{R}^{\rm new}} e^{iS[\mathbb{A}^{\rm new},0]} F[\mathbb{A}^{\rm new},0] \prod_{n=1}^3 \mathcal{D}A_n^{\rm new}\, \mathcal{D}A_0^{\rm old}}{\int_{\mathcal{R}^{\rm new}} e^{iS[\mathbb{A}^{\rm new},0]} \prod_{n=1}^3 \mathcal{D}A_n^{\rm new}\, \mathcal{D}A_0^{\rm old}} \end{aligned} \tag{8.8}$$

by making the gauge transformation (8.7). In the second expression, the integrands are independent of A_0, and the factors of $\int \mathcal{D}A_0^{\rm old}$ cancel between numerator and denominator, using the definition (8.6).

If F were not a gauge invariant operator, we could always replace it by its gauge-averaged expression,

$$\hat{F}[A] = \frac{\int d\Lambda\, F[A']}{\int d\Lambda} \tag{8.9}$$

where A' is as in (8.4), and $d\Lambda$ is the invariant group measure, satisfying

$$\int d\Lambda\, f(\Lambda) = \int d\Lambda\, f(\Lambda\Lambda_0) \tag{8.10}$$

for any SU(3) matrix Λ_0 and any function f.

Problem: If F is not gauge invariant, show that $\langle F\rangle$ as defined in (8.6) is the same as $\langle \hat{F}\rangle$.

Now the path integral is reduced to the simpler expression,

$$Z = \int e^{iS[\mathbb{A},A_0]} \mathcal{D}\mathbb{A} \tag{8.11}$$

where

$$S = \frac{1}{2g^2}\int d^4x \left(\vec{\mathbb{E}} : \vec{\mathbb{E}} - \vec{\mathbb{B}} : \vec{\mathbb{B}}\right) . \tag{8.12}$$

Since $A_0 = 0$,

$$\vec{\mathbb{E}} = -\partial_0 \vec{\mathbb{A}} = -\dot{\vec{\mathbb{A}}} \tag{8.13}$$

and $\mathbb{B}$ is just as it was before,[25]

$$\vec{\mathbb{B}} = \nabla_\times \vec{\mathbb{A}} - \tfrac{1}{2}\,\vec{\mathbb{A}} \stackrel{\times}{\times} \vec{\mathbb{A}} \tag{8.14}$$

[25] RPF had written $\vec{\mathbb{B}} = \nabla_\times \vec{\mathbb{A}} - \vec{\mathbb{A}} \stackrel{\times}{} \vec{\mathbb{A}}$. I have corrected it here and in some subsequent equations to indicate the operation needed for the spatial indices of the interaction term. The spatial cross product introduces a factor of 2 that must be compensated.

(for example, $B_z = \partial_x A_y - \partial_y A_x - [A_x, A_y]$). In terms of the gauge field, the Lagrangian is

$$L = \frac{1}{2g^2}\int d^3x \left(\dot{\vec{\mathbb{A}}} : \dot{\vec{\mathbb{A}}} - (\nabla_\times \vec{\mathbb{A}} - \tfrac{1}{2}\,\vec{\mathbb{A}} \stackrel{\times}{\times} \vec{\mathbb{A}})^2\right) . \tag{8.15}$$

This is analogous to the Lagrangian for a particle moving in a potential

$$L = \tfrac{1}{2} m\, \dot{x}^2(t) - V(x(t)) \tag{8.16}$$

where $\vec{\mathbb{A}}$ is like the position of the particle, $\dot{\vec{\mathbb{A}}} : \vec{\mathbb{A}}$ plays the role of the kinetic term, and $\vec{\mathbb{B}} : \vec{\mathbb{B}}$ is the potential. This situation is unique to the $A_0 = 0$ gauge; in other gauges we would have terms like $\dot{\vec{\mathbb{A}}} : \vec{\mathbb{A}}$, and the separation between kinetic and potential energy would no longer be so clean.

For simplicity we suppressed the source term. In $A_0 = 0$ gauge, it is

$$L_{\rm source} = \int d^3x\, \vec{\mathbb{A}} : \vec{\mathbb{J}}\,, \tag{8.17}$$

and one sees that the charge density ρ does not enter. Then the equation of motion for $\vec{\mathbb{A}}$ implied by the full Lagrangian is

$$\ddot{\vec{\mathbb{A}}} - \mathbb{D}_\times \mathbb{B} = g^2 \mathbb{J} \tag{8.18}$$

We also have the nondynamical equations

$$\mathbb{D}\cdot\mathbb{B} = 0, \quad \dot{\mathbb{B}} + \mathbb{D}_\times \mathbb{E} = 0 \tag{8.19}$$

which are identities, due to the way $\mathbb{B}$ and $\mathbb{E}$ are defined in terms of $\mathbb{A}$. Recall that we found one further equation by varying the covariant form of the Lagrangian, namely Gauss's law,

$$\mathbb{D}\cdot\mathbb{E} = g^2\rho \tag{8.20}$$

Problems. Gauss's law does not seem to arise from the path integral formulation in $A_0 = 0$ gauge. What happened to it?

What condition must the source J_μ obey in order for $\int d^4x\, \vec{J}_\mu \cdot \vec{A}_\mu$ to be gauge invariant? What is the physical significance of this condition when J_μ is the quark current?

Show that the equations of motion of ψ imply that $D_\mu J_\mu = 0$.

9. HAMILTONIAN FORMULATION OF QCD* (11-17-87)

Just as in ordinary quantum mechanics the state of a system is specified by a wavefunction $\psi(x,t)$, in the purely gluonic version of QCD we can characterize physical states by a wave functional $\Psi[\mathbb{A}(\vec{x},t)]$ that depends on the gauge field $\mathbb{A}$. (We continue to work in $A_0 = 0$

gauge.) In this lecture we shall explore the analogy somewhat, and discuss the Hamiltonian formalism for evolving Ψ in time.

Perhaps the closest analogy to the situation in field theory, where we have infinitely many dynamical variables, would be a lattice of atoms, say, interacting with each other through some potential V that depends on the positions $\vec{q}(\vec{n},t)$, where $\vec{n}$ is a lattice vector telling us which atom is being referred to. The action is

$$S = \int dt\, L = \int dt\, \left(\tfrac{1}{2} m \sum_{\vec{n}} \left| \dot{\vec{q}}(\vec{n},t) \right|^2 - V\left(\vec{q}(\vec{n},t)\right) \right) . \tag{9.1}$$

Now if $\psi_i(\vec{q}(\vec{n}))$ is the wave function at some initial time t_i, then the amplitude for reaching a state $\psi_f(\vec{q}(\vec{n}))$ at a later time t_f is given by an ordinary integral

$$\int \prod_{\vec{n}} d^3 q_i(\vec{n})\, d^3 q_f(\vec{n}) \tag{9.2}$$
$$\psi_f(\vec{q}_f(\vec{n}))\, K\Big(\vec{q}_f(\vec{n}), t_f; \vec{q}_i(\vec{n}), t_i\Big)\, \psi_i(\vec{q}_i(\vec{n}))$$

and the function K that propagates the initial state is given by a path integral

$$K\Big(\vec{q}_f(\vec{n}), t_f; \vec{q}_i(\vec{n}), t_i\Big) = \prod_{\vec{n}} \mathcal{D}\vec{q}(\vec{n},t)\, e^{iS[\vec{q}(\vec{n},t)]} \tag{9.3}$$

where the integral is over functions $\vec{q}(\vec{n},t)$ satisfying

$$\begin{aligned} \vec{q}(\vec{n},t_i) &= \vec{q}_i(\vec{n})\,, \\ \vec{q}(\vec{n},t_f) &= \vec{q}_f(\vec{n})\,. \end{aligned} \tag{9.4}$$

In the same way, we can assign to each state in QCD a wave functional $\Psi[\mathbb{A}(\vec{x},t)]$ such that $|\Psi|^2$ is the probability density for the gauge field to have the value $\mathbb{A}(\vec{x},t)$, for each point in space, at a given time. Here $\mathbb{A}$ corresponds to $\vec{q}$, and the position $\vec{x}$ corresponds to the lattice vector $\vec{n}$ in the atomic crystal analog. The kernel for time evolution of $\Psi[\mathbb{A}]$ was given in (8.2), which is the analog of (9.3).

Alternatively, the time evolution of Ψ can be described in differential rather than integral form—Schrödinger's equation! To do this, we must first find the Hamiltonian. In the finite system (9.1), the canonically conjugate momenta are

$$\vec{p}(\vec{n}) = \frac{\partial L}{\partial \dot{\vec{q}}(\vec{n})} = m\, \dot{\vec{q}}(\vec{n}) \tag{9.5}$$

They can be represented by

$$\vec{p}(\vec{n}) = \frac{1}{i}\, \frac{\partial}{\partial \vec{q}(\vec{n})} \tag{9.6}$$

(taking $\hbar = 1$) since $-i\partial/\partial\vec{q}$ has the same commutation relation with $\vec{q}$ as $\vec{p}$ has canonically. Similarly in QCD the momentum conjugate to $A^a_n(\vec{x})$ is

$$p^a_n(\vec{x}) = \frac{\delta L}{\delta \dot{A}^a_n(\vec{x})} = \dot{A}^a_N(\vec{x}) = -E^a_n(\vec{x}) \tag{9.7}$$

where we used the Lagrangian (8.15) after rescaling $\mathbb{A} \to g\mathbb{A}$, and the operation that appears is a *functional* derivative,

$$\frac{\delta A^a_n(\vec{x})}{\delta A^b_m(\vec{x}')} = \delta_{ab}\, \delta_{nm}\, \delta^{(3)}(\vec{x} - \vec{x}') \tag{9.8}$$

This is the natural generalization of the partial derivative to the case of infinitely many variables, labeled by a continuous index $\vec{x}$. Notice that $\vec{p}$ is just minus the color electric field. It can also be written as

$$p^a_n(\vec{x}) = \frac{1}{i}\, \frac{\delta}{\delta A^a_n(\vec{x})} \tag{9.9}$$

similarly to (9.6). Now the Hamiltonian can be constructed. For a discrete system like the lattice, it is

$$H = \left(\sum_{\vec{n}} \vec{p}(\vec{n}) \cdot \dot{\vec{q}}(\vec{n}) - L \right)_{\dot{\vec{q}} = \vec{p}/m} . \tag{9.10}$$

For QCD, one simply replaces the sum with an integral, so that

$$\begin{aligned} H &= \tfrac{1}{2} \int d^3x \left(\vec{\mathbb{E}}^2 + \vec{\mathbb{B}}^2 \right) \\ &= \tfrac{1}{2} \int d^3x \left(-\left(\frac{\delta}{\delta \vec{\mathbb{A}}} \right)^2 + \vec{\mathbb{B}}^2 \right) . \end{aligned} \tag{9.11}$$

Then the Schrödinger equation for $\Psi[\mathbb{A}]$ is

$$\frac{1}{i}\, \frac{\partial}{\partial t} \Psi = H \Psi\,. \tag{9.12}$$

Previously we noted that three of the four Maxwell equations of QCD emerged from the gauge-fixed Lagrangian (8.15), and the definitions of $\mathbb{E}$ and $\mathbb{B}$, but Gauss's law,

$$\mathbb{D}.\vec{\mathbb{E}} = g^2 \vec{\rho},$$

did not appear. However, the *time derivative* of Gauss's law can be deduced as follows:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\mathbb{D}.\vec{\mathbb{E}} \right) &= -\frac{\partial}{\partial t} \left(\nabla.\, \dot{\vec{\mathbb{A}}} - i \vec{\mathbb{A}} \times \dot{\vec{\mathbb{A}}} \right) \\ &= -\nabla.\, \ddot{\vec{\mathbb{A}}} + i \vec{\mathbb{A}} \times \ddot{\vec{\mathbb{A}}} \\ &= \mathbb{D}.\, \ddot{\vec{\mathbb{A}}} \end{aligned} \tag{9.13}$$

Using the equation of motion for $\mathbb{A}$, eq. (8.18), this becomes

$$\frac{\partial}{\partial t} \left(\mathbb{D}.\vec{\mathbb{E}} \right) = -\mathbb{D}.\, (\mathbb{D} \times \vec{\mathbb{B}}) - g^2 \mathbb{D}.\vec{\mathbb{J}} \tag{9.14}$$

The middle term would vanish trivially if $\mathbb{D}$ was the ordinary gradient, but since the components of $\mathbb{D}$ do not commute, more care is required. One finds that $\mathbb{D}.\, (\mathbb{D} \times \vec{\mathbb{B}})$ is

$$[D_x, D_y] B_z \text{ plus cyclic permutations.}$$

But this is just $i[F_{xy}, B_z]$, as you showed in a previous exercise, which vanishes because $F_{xy} = B_z$. Furthermore the source is covariantly conserved,

$$D_\mu J_\mu = 0,$$

so that

$$\mathbb{D}.\,\vec{\mathbb{J}} = -\frac{\partial}{\partial t}\vec{\rho}$$

in $A_0 = 0$ gauge. (In our conventions, $A_\mu B_\mu = -A_0 B_0 + A_i B_i$, and $\rho = -J_0$.[26]) Therefore the condition

$$\frac{\partial}{\partial t}(\mathbb{D}.\,\vec{\mathbb{E}}) = g^2 \frac{\partial}{\partial t}\vec{\rho} \tag{9.15}$$

is a consequence of the equations of motion. Consequently, if the wave functional satisfied

$$\left[\mathbb{D}.\,\vec{\mathbb{E}} - g^2\vec{\rho}\right]\Psi[\mathbb{A}(x)] = 0 \tag{9.16}$$

at some initial time, it would continue to do so forever. Therefore Gauss's law can be implemented by imposing it as a constraint on the state of the system, Ψ.

Notice that (9.16) is a functional differential equation, since $\mathbb{E}$ is to be interpreted as $-i\delta/\delta\mathbb{A}$. Moreover (9.16) is an infinite set of constraints, one at each point in space. One might wonder whether solutions exist, since the constraint operator

$$\vec{C}(\vec{x}) \equiv \mathbb{D}.\,\vec{\mathbb{E}}(\vec{x}) - g^2\vec{\rho}(\vec{x}) \tag{9.17}$$

does not commute at different positions,

$$[\vec{C}(\vec{x}), \vec{C}(\vec{x}')] \neq 0\,.$$

To be consistent, we require that this new operator also annihilates the wave functional. If the commutator is a linear combination of $\vec{C}$'s there is no problem, but if not we might generate more and more constraints, to the point that no solution existed. It turns out to be nicer to investigate this not with the $\vec{C}$'s directly, but rather with their weighted averages, defined by

$$\Gamma(\mu) \equiv \int \mu^i(\vec{x})\, C^i(\vec{x})\, d^3x\,. \tag{9.18}$$

One can show that

$$[\Gamma(\mu), \Gamma(\nu)] = \Gamma(\lambda) \tag{9.19}$$

where

$$\vec{\lambda} = \vec{\mu} \times \vec{\nu} \tag{9.20}$$

[26] This choice of the metric signature is not consistent throughout the lectures.

Thus the commutators produce no new constraints; instead they form a closed algebra. It is the algebra of the color group SU(3), for we could define generators of local color transformations

$$\hat{\Gamma}(\mu) = \mu^i(x) T^i$$

in the three-dimensional (*i.e. fundamental*) representation of SU(3), and they would satisfy the same relations (9.19,9.20) as the $\Gamma(\mu)$.

Problem. Prove eqs. (9.19,9.20).

It is therefore not surprising that the Γ operators generate gauge transformation on the state Ψ. That is,

$$e^{i\Gamma(\mu)}\Psi[\mathbb{A}] = \Psi[\mathbb{A}'] \tag{9.21}$$

where $\mathbb{A}'$ is the gauge field obtained from $\mathbb{A}$ by transforming with the matrix

$$\Lambda(\vec{x}) = e^{i\vec{\mu}(\vec{x})\cdot\vec{T}} \tag{9.22}$$

The alert reader may wonder how it is possible to do gauge transformations, since we have already fixed the gauge to $A_0 = 0$. However the transformation (9.22) is time-independent, so $\Lambda^\dagger \partial_0 \Lambda = 0$ and any such Λ will keep $A_0 = 0$. Hence the Gauss's law constraint on Ψ means that Ψ must be *invariant* under the residual gauge transformations that preserve the $A_0 = 0$ condition. We can prove this directly from the equation itself: let

$$\left[\int \vec{\alpha}\cdot\left(\mathbb{D}.\,\vec{\mathbb{E}} - g^2\vec{\rho}(\vec{x})\right) d^3x\right]\Psi[\mathbb{A}] = 0 \tag{9.23}$$

for some $\vec{\alpha}(\vec{x})$. For simplicity suppose that there are no quarks, so that $\rho = 0$. Then (9.23) can be rewritten as

$$\left[\int \mathbb{D}\vec{\alpha}(\vec{x}) : \vec{\mathbb{E}}(\vec{x})\, d^3x\right]\Psi[\mathbb{A}] = 0$$

where we have integrated by parts. (The reader should satisfy himself that partial integration works for covariant derivatives.) Using the operator form of $\mathbb{E} = -i\delta/\delta\mathbb{A}$, this becomes

$$\int \mathbb{D}\vec{\alpha}(\vec{x}) : \frac{\delta\Psi}{\delta\vec{\mathbb{A}}}\, d^3x \;= 0 \tag{9.24}$$

Now the functional version of Taylor's theorem says that

$$\Psi[\vec{\mathbb{A}}(x) + \vec{\mathbb{a}}(x)] = \Psi[\vec{\mathbb{A}}(x)] + \int \vec{\mathbb{a}} : \frac{\delta\Psi[\vec{\mathbb{A}}(x)]}{\delta\vec{\mathbb{A}}(x)}\, d^3x \tag{9.25}$$

to first order in $\vec{\mathbb{a}}$. So for infinitesimal $\vec{\alpha}(\vec{x})$, Gauss's law is equivalent to

$$\Psi[\vec{\mathbb{A}}(x) + \mathbb{D}\vec{\alpha}(x)] = \Psi[\vec{\mathbb{A}}(x)] \tag{9.26}$$

which is just the result claimed, since $\mathbb{D}\vec{\alpha}(x)$ is the effect of an infinitesimal gauge transformation.

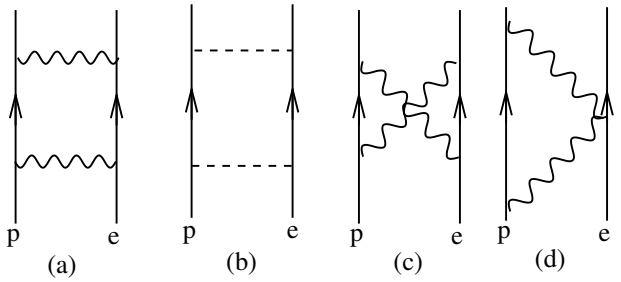


FIG. 22: Diagrams for the hydrogen atom.

However not all gauge transformations can be built up from infinitesimal ones. This is most easily demonstrated for the SU(2) subgroup of SU(3) generated by

$$\vec{\sigma} = (\lambda_1, \lambda_2, \lambda_3)$$

Imagine that we have cut off the infinite volume of space by introducing a large radius R. It is easy to check that the matrix

$$\Lambda = \sqrt{1 - \frac{x^2}{R^2}} + i\frac{\vec{x}\cdot\vec{\sigma}}{R} \tag{9.27}$$

is unitary and has determinant 1, yet cannot be written in the form of $\exp(i\vec{\alpha}^{\star}\vec{\lambda})$. We should therefore require that $\Psi[\mathbb{A}]$ be invariant under all such "large" gauge transformations, as well as the ordinary ones.

10. PERTURBATION THEORY (11-19-87)

Now we must confront the question, how do we calculate anything quantitatively in this theory? Perturbation theory (P.T.) is not very useful for bound state properties since the coupling g is large. On the other hand, we have some experience with nonperturbative processes even in nonrelativistic quantum mechanics; the hydrogen atom is not a perturbative problem—it is an exact nonperturbative solution to Schrödinger's equation. But it is nevertheless made more accurate by the smallness of the coupling e. Consider fig. 22(a). Since photon exchange is relatively infrequent, we can replace the photon exchanges by instantaneous effective interactions depicted in fig. 22(b). The Schrödinger equation takes an initial state of the proton and the electron and propagates them freely via $-(\partial/\partial x)^2$ plus interactions $V(x)$. This only works if the coupling constant is small; otherwise diagrams like fig. 22(c,d) become too important and we would have to find some other kind of effective interaction potential to represent their effect.

But there are some processes for which perturbation theory in QCD works relatively well, for example $p\bar{p}$ collisions (fig. 23). At sufficiently high energies, only one gluon might be exchanged, similarly to fig. 21. One can then predict the dynamics of the jets rather precisely using perturbation theory.

10.1. Review of P.T. from the path integral

Recall the massless scalar field theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\dot{\phi}^2 - (\nabla\phi)^2\right) + \frac{g}{3!}\,\phi^3 - S(x)\,\phi$$

The equation of motion is

$$\ddot{\phi} - \nabla^2\phi \equiv \Box\phi = \frac{g}{2}\phi^2 - S \tag{10.1}$$

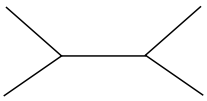
Let us ignore interactions for the moment and focus on the source term. Going to Fourier space, the equation of motion becomes

$$(\omega^2 - \vec{k}^2)\phi_k = k^2\phi_k = S_k \tag{10.2}$$

To solve for ϕ_k, a prescription must be given for treating the pole of the propagator, *i.e.,* we must add $i\epsilon$,[27]

$$\phi_k = \frac{S_k}{k^2 + i\epsilon} \tag{10.3}$$

Putting back the interaction, we have the rule that the amplitude for three ϕ particles to interact is g. Then for example the s-channel scattering diagram is



$$= \frac{g^2}{k^2 + i\epsilon} \tag{10.4}$$

Let's now compare this to the case of QCD in $A_0 = 0$ gauge, where the propagator comes from the free Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\dot{\vec{\mathbb{A}}} : \dot{\vec{\mathbb{A}}} - (\mathbb{\nabla}_{\times}\vec{\mathbb{A}})^2\right) . \tag{10.5}$$

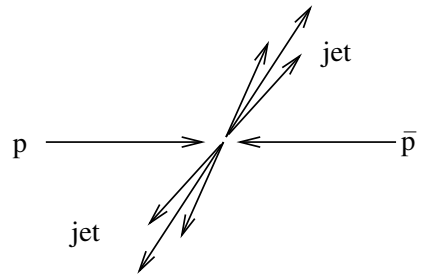


FIG. 23: Electron-positron annihilation into hadronic jets.

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[27] Throughout these lectures, RPF avoids writing factors of i for the vertices and the propagators. Later, he will claim that the only factors of i that are necessary to keep track of can be associated with loops. I have not checked whether his rules for the cubic and quartic gluon interactions give rise to the correct sign of interference for the diagrams contributing to four-gluon scattering, but his later claim implies that he did so.

Here we have rescaled $\mathbb{A} \to g\mathbb{A}$ to get the coupling out of the propagator and back into the interactions. Including a classical source $\vec{\mathbb{S}}$, the equation of motion is

$$\ddot{\vec{\mathbb{A}}} - \nabla \times (\nabla \times \vec{\mathbb{A}}) = -\vec{\mathbb{S}} \tag{10.6}$$

or equivalently

$$\ddot{\mathbb{A}}^i - \nabla^2 \mathbb{A}^i + \nabla(\nabla \cdot \mathbb{A}^i) = -\mathbb{S}^i \tag{10.7}$$

Hence, going to Fourier space,

$$(\omega^2 - \mathbb{k}^2)\,\mathbb{A} + \mathbb{k}(\mathbb{k} \cdot \mathbb{A}) = \mathbb{S} \tag{10.8}$$

(omitting the gauge index i and subscript k for brevity). Next dot $\mathbb{k}$ into this equation to get

$$(\omega^2 - \mathbb{k}^2)\,\mathbb{k} \cdot \mathbb{A} + \mathbb{k}^2\,\mathbb{k} \cdot \mathbb{A} = \omega^2\,\mathbb{k} \cdot \mathbb{A} = \mathbb{k} \cdot \mathbb{S}\,. \tag{10.9}$$

We can therefore eliminate $\mathbb{k} \cdot \mathbb{A}$ from eq. (10.8) and find

$$k^2\,\mathbb{A} = \frac{1}{\omega^2}\left(\mathbb{S} - \mathbb{k}(\mathbb{k} \cdot \mathbb{S})\right)\,. \tag{10.10}$$

In analogy to the scalar field example, one can read off the propagator

$$\mathbb{P}(k) = \frac{\mathbb{1} - \mathbb{k}\mathbb{k}/\omega^2}{k^2 + i\epsilon} \tag{10.11}$$

in the form of a 3×3 matrix for the spatial components of the gauge field, with the understanding that $\mathbb{k}\mathbb{k}$ denotes the outer product of the spatial momenta components.

10.2. Perturbation theory for QCD

Now we would like to perturbatively compute path integrals involving gauge invariant functionals $F[\mathbb{A}, 0]$ (showing explicitly that A_0 is set to zero),

$$\langle F \rangle = \frac{\int e^{iS[\mathbb{A},0]}\,F[\mathbb{A},0]\,\mathcal{D}\mathbb{A}}{\int e^{iS[\mathbb{A},0]}\,\mathcal{D}\mathbb{A}} \tag{10.12}$$

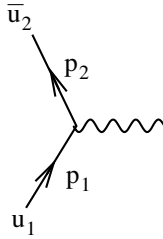
where the action is

$$S[\mathbb{A},0] = \tfrac{1}{2}\int \left[\dot{\vec{\mathbb{A}}} : \dot{\vec{\mathbb{A}}} - \vec{\mathbb{B}} : \vec{\mathbb{B}}\right]\,d^4x + g\int \vec{\mathbb{A}} : \vec{\mathbb{J}}\,d^4x \tag{10.13}$$

with $\vec{\mathbb{B}} = \nabla_\times \vec{\mathbb{A}} - (g/2)\,\vec{\mathbb{A}} \overset{\times}{\times} \vec{\mathbb{A}}$ and the current

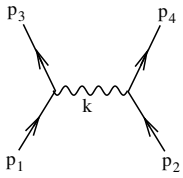
$$\mathbb{J}^i_n = \bar{\psi}(\vec{x},t)\,\gamma_n\,\frac{\lambda^i}{2}\,\psi(\vec{x},t) \qquad (n = 1,2,3)\,. \tag{10.14}$$

The amplitude for a quark-gluon interaction can be written as



$$= \sqrt{4\pi g^2}\sum_{n,i}\left(\bar{u}_2\gamma_n\,\frac{\lambda^i}{2}\,u_1\right)a^i_n \tag{10.15}$$

where a^i_n is the combined spin-color polarization of the gluon and the coupling is rescaled so that the Coulomb interaction is g^2/r instead of $g^2/(4\pi r)$. Therefore the gluon-exchange diagram giving rise to the potential is



$$= \pi g^2\left(\bar{u}_3\gamma_n\lambda^i u_1\right) \times \left(\bar{u}_4\gamma_m\lambda^j u_2\right)\left(\frac{\delta_{nm} - \frac{k_m k_n}{\omega^2}}{k^2}\right)\delta_{ij} \tag{10.16}$$

This can be put into a simpler form using the conservation of the quark currents, *e.g.*,

$$\begin{aligned} k_\mu(\bar{u}_3\gamma_\mu\lambda^i u_1) &= \bar{u}_3(\not{p}_1 - \not{p}_3)\lambda^i u_1 \\ &= \bar{u}_3(m-m)\lambda^i u_1 = 0 \end{aligned} \tag{10.17}$$

since the gluon vertex conserves flavors. Therefore $k_n\bar{u}_3\gamma_n\lambda^i u_1 = \omega\,\bar{u}_3\gamma_0\lambda^i u_1$ and we can rewrite the right-hand side of (10.16) as[28]

$$\pi g^2\left(\bar{u}_3\gamma_\mu\lambda^i u_1\right)\left(\frac{-\delta_{\mu\nu}}{k^2}\right)\delta_{ij}\left(\bar{u}_4\gamma_\nu\lambda^j u_2\right) \tag{10.18}$$

in which the gluon propagator takes a Lorentz covariant form. This is an illustration of the fact that physical amplitudes are independent of the choice of gauge. Nevertheless we must make *some* choice. Consider the gluon Lagrangian with no choice of gauge imposed,

$$\mathcal{L} \sim \left[\partial_\mu A_\nu - \partial_\nu A_\mu - A_\mu \overset{\times}{} A_\nu\right]^2 \tag{10.19}$$

The noninteracting part has the structure $k^2\delta_{\mu\nu} - k_\mu k_\nu$ in momentum space, which is noninvertible. Hence we need to fix the gauge to define the gluon propagator.

Another interesting observation is that when varying the full action, the equation of motion takes the form

$$\begin{aligned} \Box A_\nu - \partial_\nu\partial_\mu A_\mu &= S_\nu \\ \implies k^2 A_\nu - k_\nu(k\cdot A) &= S_\nu(k) \\ \implies 0 &= k\cdot S \end{aligned} \tag{10.20}$$

where the source S_ν now includes contributions that are nonlinear in A for nonvanishing background gauge fields, in addition to the quark current contribution. In this case, current conservation is more complicated than for QED, where the current comes only from the charged fermions.

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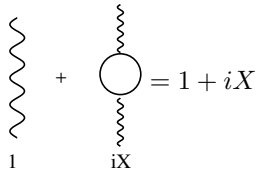
[28] RPF writes $\delta_{\mu\nu}$ for the Minkowski metric tensor, even though he is not working in Euclidean space. Moreover he normally does not distinguish between covariant and contravariant Lorentz indices.

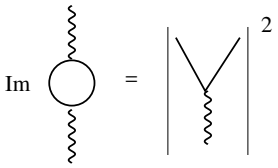
10.3. Unitarity

Consider the lowest order processes contributing to gluon propagation (postponing for the moment the issue of gluon loops):[29]



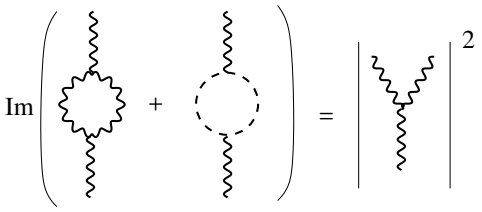
$$= 1 + iX \tag{10.21}$$

The probability associated with this amplitude is $1 + i(X - X^*) + \ldots$. This means that $i(X - X^*)$ is the probability of *not* producing a gluon from a gluon, which is the probability of instead producing a $q\bar{q}$ pair:



(10.22)

However if we try to do the same thing with gluons in the loop, and using a covariant gauge for the gluon propagator, the analogous relation breaks down. To fix it, we need to add ghost fields,



(10.23)

The ghost fields (dashed line) do not appear on the right-hand side of eq. (10.23) because ghosts never enter into a final state. Alternatively, we can work in $A_0 = 0$ gauge instead of covariant gauge, and dispense with the ghosts.

Before Faddeev and Popov solved the problem of consistently incorporating ghosts at any order of perturbation theory, I was trying to figure out how to make it work beyond one loop, without success. At that time it did not occur to me to use $A_0 = 0$ gauge, because of a prejudice based on experience with QED. People originally tried to formulate QED in a nonrelativistic gauge, $\mathbb{\nabla} \cdot \mathbb{A} = 0$, but nobody understood how to renormalize this theory in a relativistic way.

Another way to try to get around the gauge fixing problem is to temporarily give the gluons a mass. Then the propagator exists, and one can try to take $m_g \to 0$ at the end of the calculation. However for processes beyond tree level, this limit turns out not to exist.

[29] The factor of i associated with the loop will be explained at the end of the lecture.

10.4. Gluon self-interactions

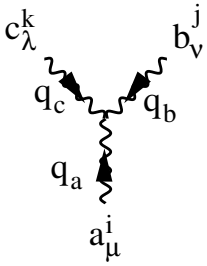
For now we will avoid the ghosts by continuing in $A_0 = 0$ gauge. The next step is to write the rules for the gluon self-interaction vertices. We continue to write combined spin/color polarization vectors $a, b, c, \ldots$ corresponding to a plane wave solution $A^i_\nu = (a^i_\nu e^{iq^\mu_a x_\mu} + b^i_\nu e^{iq^\mu_b x_\mu} + c^i_\nu e^{iq^\mu_c x_\mu}$, and taking all q_i to point inwards toward the vertex, so that $\sum_i q^\mu_i = 0$. Recall that the cubic interaction Lagrangian is

$$g(\partial_\mu A_\nu - \partial_\nu A_\mu) A_\mu{}^\times A_\nu \tag{10.24}$$

By substituting the plane wave solution for the fields, we can read off the rule for the 3-gluon vertex,

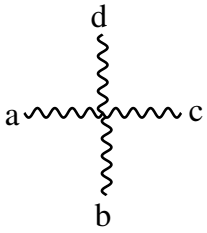
$$g(q^\mu_a a_\nu - q^\nu_b a_\mu) \cdot (b_\mu{}^\times c_\nu) + \text{ cyclic permutations} \tag{10.25}$$

The permutations can be reorganized into the form



$$= \sqrt{4\pi g^2} \begin{pmatrix} (q^a - q^c)_\nu \left(b_\nu \cdot (a_\mu{}^\times c_\mu)\right) \\ +(q^b - q^a)_\nu \left(c_\nu \cdot (b_\mu{}^\times a_\mu)\right) \\ +(q^c - q^b)_\nu \left(a_\nu \cdot (c_\mu{}^\times b_\mu)\right) \end{pmatrix} \tag{10.26}$$

after rescaling the coupling as before. Similarly, for the four-gluon amplitude, we obtain



$$= \frac{4\pi g^2}{4} (a_\mu{}^\times b_\nu) \cdot (c_\mu{}^\times d_\nu) + \text{symmetric permutations} \tag{10.27}$$

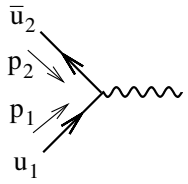
For completeness, the gluon propagator is again

$$= \frac{\delta_{mn} - k_m k_n/\omega^2}{k^2} \delta_{ij} \tag{10.28}$$

and the rules for quarks are

$$= \frac{\delta_{ij}}{\not{p} - m_q} \tag{10.29}$$

and[30]



$$= \sqrt{4\pi g^2}\,\bar{u}_2(\,\vec{a}_\mu\cdot\vec{\lambda}\,\gamma_\mu)u_1 \qquad (10.30)$$

We can rewrite the gluon propagator in a more covariant-looking form by introducing the 4-vector $\eta_\mu = (1,0,0,0)$. Taking $q_\mu = (\omega, \vec{k})$, we have $\omega = \eta\cdot q$ and $(0,\vec{k}) = q_\mu - \eta_\mu(\eta\cdot q)$, so

$$\frac{k_n k_m}{\omega^2} = \frac{(q_\mu - \eta_\mu(\eta\cdot q))\,(q_\nu - \eta_\nu(\eta\cdot q))}{(\eta\cdot q)^2} \qquad (10.31)$$

Moreover $-\delta_{mn} = \delta_{\mu\nu} - \eta_\mu\eta_\nu$. The gluon propagator (ignoring color indices) becomes

$$P_{\mu\nu} = \frac{-\delta_{\mu\nu} + \frac{q_\mu\eta_\nu + q_\nu\eta_\mu}{q\cdot\eta} - \frac{q_\mu q_\nu \eta^2}{(\eta\cdot q)^2}}{q^2 + i\epsilon} \qquad (10.32)$$

which has the property $P_{\mu\nu}\eta_\nu = 0$. All of the η-dependence must drop out of physical amplitudes for them to be Lorentz invariant. We will see in section 12 that the $q_\mu q_\nu$ term can be eliminated by an appropriate gauge fixing procedure.

10.5. Loops

Thus far all the rules have been unencumbered by any factors of i. We can consistently push them into the rules for loops, integrating over the internal momentum of the loop,

$$\begin{aligned} i\int\frac{d^4p}{(2\pi)^4} \quad &\text{for gluon loops} \\ -i\int\frac{d^4p}{(2\pi)^4} \quad &\text{for quark loops} \end{aligned} \qquad (10.33)$$

These are the only factors of i that one ever needs.[31]

[32] The minus sign for fermion loops can be understood as the result of doing a 360° rotation, illustrated by taking the two ends of a belt and exchanging their positions

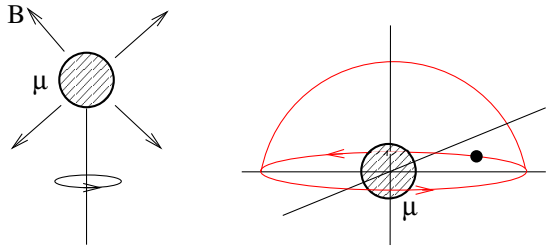
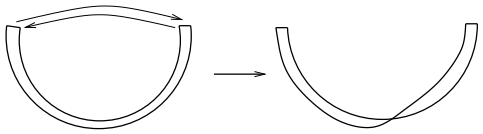
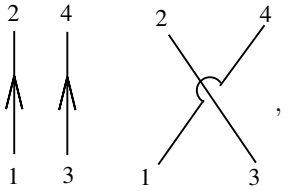


FIG. 24: Left: magnetic monopole with a charge transported around its Dirac string. Right: the charge is transported around the monopole within the plane.

while keeping the orientations of ends of the belts fixed. Although each fermion by itself undergoes only a 180° rotation, relative to each other it is 360°, which as we know for fermions introduces a relative sign, symbolized by the kink in the belt.

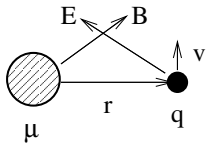


Similarly when we exchange the positions of two fermions in diagrams such as



they differ from each other by a minus sign because of Fermi statistics. There is inherently a 360° rotation of their relative orientations. A similar exchange occurs when there is a fermion loop, leading to the minus sign in the diagrammatic rule.

An interesting aside illustrates the origin of this sign for a composite system that behaves like a fermion. This is the combination of a magnetic monopole of strength μ and an electrically charged scalar, with charge q, separated by some distance, $\vec{r}$ [33]



Even in the absence of any relative motion between the two constituents, this system has angular momentum in

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[30] RPF aligns $\vec{p}_2$ against the flow of fermion number since the spinor is $\bar{u}_2$, not $\bar{v}_2$. We will see this again in section 11. The point of defining final state momenta as being negative is to make the Mandelstam variables all look the same, all involving plus signs rather than minus signs.

[31] See note 27

[32] The following material appears in my notes as an interruption of the new subject that has just been introduced, section 11, as though it suddenly occurred to RPF that he had meant to discuss it earlier. In characteristic showman fashion, he removed his own belt to do the demonstration.

[33] Characteristically, RPF does not call this by its common name, dyon. Several pages of his personal notes are devoted to this problem.

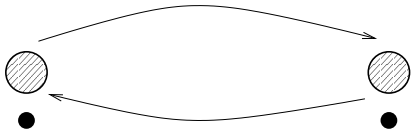


FIG. 25: Interchange of two monopole-charge systems.

the $\vec{r}$ direction, which can be deduced by imagining that we try to move the charge with velocity $\vec{v}$ as shown. Since q is moving in a magnetic field, it experiences a force that gives a torque on the system, as if it were a gyroscope, showing that it has angular momentum.

Since $\mathbb{B} = \mathbb{\nabla} \times \mathbb{A}$, $\mathbb{B}$ can never have a divergence unless there is a Dirac string. If μ is quantized such that $\mu q = \hbar/2$, then the change in phase of an electron as it moves around the string is $e^{ie \int \vec{A} \cdot \vec{dx}}$, which is -1 if μ is quantized properly. This is easiest to see in the right-hand part of fig. 24 by considering the phase change of a charge moving in the plane of the monopole using Green's theorem,

$$\begin{aligned} e^{iq \int A \cdot dx} &= e^{iq\Phi} \\ &= e^{iq(\frac{1}{2} 4\pi\mu)} = e^{iq2\pi\mu} \\ &= e^{i\pi} \end{aligned} \tag{10.34}$$

where Φ is the flux of B through the upper hemisphere. But we have merely rotated the system by 360°, so this phase change shows that it behaves like a particle of spin 1/2.

Moreover if we consider two such systems such as in fig. 25, and interchange them, their combined wave function acquires a phase of -1. It comes from the combined phase changes of the charges as they move around the opposite monopole by 180°.

11. SCATTERING PROCESSES (11-24-87)

Let us now consider the scattering of two quarks as shown in fig. 26.[34] This can be measured by doing p-p scattering, since the parton model allows us to relate the two processes. The parton distribution functions are measured by deep inelastic scattering experiments.

Most of the time the scattering does not produce jets, but these are the observables we are interested in. The scattered quarks determine the directions of the jets. This part of the problem—how quarks hadronize into jets—can be understood from the observations of $e^+e^- \to$ [2 hadron jets] through the electromagnetic process shown in fig. 27.

Of course it is also possible to get jets originating from gluons produced in the QCD scattering process. This

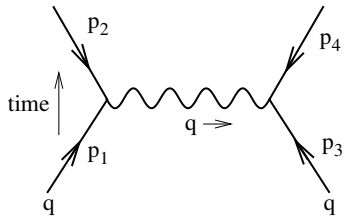


FIG. 26: Quark-quark scattering.

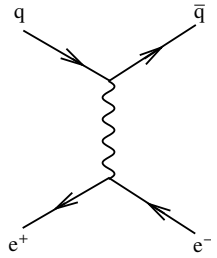


FIG. 27: Electromagnetic production of jets.

has to be taken into account, but for simplicity we will start with the quark production process. Our goal is to measure α_s through scattering. Historically this analysis helped to design the experiments observing these processes, and QCD helped to tune the phenomenological models needed to make the predictions.

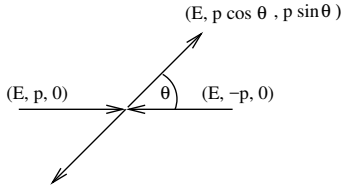
Recall the gluon propagator (10.32) in $\eta \cdot A = 0$ gauge. The terms involving η drop out of the amplitude because of the conservation of the external quark currents. Then the amplitude becomes

$$T = [4\pi]\, g^2 \left(\bar{u}_2 \gamma_\mu \frac{\lambda^i}{2} u_1 \right) \frac{1}{q^2} \left(\bar{u}_4 \gamma_\mu \frac{\lambda^i}{2} u_3 \right) \tag{11.1}$$

Conventionally we express it in terms of the Mandelstam variables. Neglect quark masses, since we want high enough energies to get clean jets. Then

$$\begin{aligned} s &= (p_1 + p_3)^2 = 4E^2 \text{ in c.m. frame} \\ t &= (p_1 + p_2)^2 = -(\text{momentum transfer})^2 \\ u &= (p_1 + p_4)^2 \end{aligned} \tag{11.2}$$

which have the property $s + t + u = \sum_{i=1}^{4} m_i^2$ (left as an exercise for the reader to prove). In the center-of-mass frame,



the momentum transfer is given by

$$\begin{aligned} t &= (0,\, p(1 - \cos\theta),\, p \sin\theta)^2 \\ &= -2p^2(1 - \cos\theta) = q^2 \end{aligned} \tag{11.3}$$

[34] The unconventional choice of momentum labels is deliberate; see note 30.

Next we need $|T|^2$, which depends upon the polarizations of the quarks. If these are not measured, then we are only interested in

$$\overline{|T|^2} = \sum_{\substack{\text{init. spins}\\ \&\text{ colors}}} \sum_{\substack{\text{final spins}\\ \&\text{ colors}}} \frac{|T|^2}{\left(\begin{array}{c}\text{no. of spins \&}\\ \text{colors in init. state}\end{array}\right)} \qquad (11.4)$$

If we measure the final state polarizations but the incoming beams are unpolarized, then we should omit the sum over final state spins.

In eq. (11.1) we have implicitly assumed that the spinors u_i are products of spin and color factors, which we can write as

$$u_1 = U_1\,\alpha_1,\ \ etc. \qquad (11.5)$$

Then

$$\begin{aligned} |T|^2 &= \left(\frac{4\pi g^2}{q^2}\right)^2 \left[\bar U_2\bar\alpha_2\gamma_\mu\frac{\lambda^i}{2}U_1\alpha_1\right]\left[\bar U_4\bar\alpha_4\gamma_\mu\frac{\lambda^i}{2}U_3\alpha_3\right] \\ &\times \left(\left[\bar U_2\bar\alpha_2\gamma_\nu\frac{\lambda^j}{2}U_1\alpha_1\right]\left[\bar U_4\bar\alpha_4\gamma_\nu\frac{\lambda^j}{2}U_3\alpha_3\right]\right)^* \\ &\to \sum_{\text{spins}}\left(\frac{4\pi g^2}{q^2}\right)^2 \qquad (11.6) \\ &\times \left(\bar U_2\gamma_\mu U_1\right)\left(\bar U_4\gamma_\mu U_3\right)\Big(\left(\bar U_2\gamma_\nu U_1\right)\left(\bar U_4\gamma_\nu U_3\right)\Big)^* \\ &\times \left(\bar\alpha_2\frac{\lambda^i}{2}\alpha_1\right)\left(\bar\alpha_4\frac{\lambda^i}{2}\alpha_3\right)\left(\bar\alpha_2\frac{\lambda^j}{2}\alpha_1\right)^*\left(\bar\alpha_4\frac{\lambda^j}{2}\alpha_3\right)^* \end{aligned}$$

The sum on spins gives, for example,

$$u_2\bar u_2 = \not p_2 + m \qquad (11.7)$$

leading to the traces

$$\mathrm{tr}\Big(\gamma_\nu(\not p_2+m)\gamma_\mu(\not p_1+m)\Big)\mathrm{tr}\Big((\not p_4+m)\gamma_\mu(\not p_3+m)\gamma_\nu\Big)$$

Similarly, the sum on colors gives

$$\sum_{\text{colors}}\left(\bar\alpha_2\frac{\lambda^i}{2}\alpha_1\bar\alpha_1\frac{\lambda^j}{2}\alpha_2\right) = \mathrm{tr}\frac{\lambda^i}{2}\frac{\lambda^j}{2} = \frac{1}{2}\delta_{ij} \qquad (11.8)$$

Recall that

$$\begin{aligned} \mathrm{tr}\,\gamma_\nu\not p_2\gamma_\mu\not p_1 &= 4\Big(p_{2\nu}p_{1\mu} + p_{2\mu}p_{1\nu} - (p_1{\cdot}p_2)\delta_{\mu\nu}\Big) \\ \mathrm{tr}\,\not p_4\gamma_\mu\not p_3\gamma_\nu &= 4\Big(p_{4\nu}p_{3\mu} + p_{4\mu}p_{3\nu} - (p_3{\cdot}p_4)\delta_{\mu\nu}\Big) \end{aligned}$$

neglecting masses. The total color factor is

$$\frac{\delta_{ij}\delta_{ij}}{4} = \frac{8}{4} = \frac{N^2-1}{4} \qquad (11.9)$$

where we indicated the more general result for SU(N) at the end, and we have not yet included the $1/N^2$ from averaging over the quark colors. Putting everything together,

$$\begin{aligned} \overline{|T|^2} &= \left(\frac{(4\pi g^2)^2}{2^2N^2q^4}\right)16{\cdot}\tfrac14(N^2-1) \\ &\times\ [2(p_2{\cdot}p_4)(p_1{\cdot}p_3) + 2(p_1{\cdot}p_4)(p_1{\cdot}p_3)] \\ &= \frac{N^2-1}{4N^2}\left(\frac{16}{4}\right)\frac{(4\pi g^2)^2}{t^2}\tfrac12(s^2+u^2) \qquad (11.10) \end{aligned}$$

and the differential cross section is[35]

$$\frac{d\sigma}{dt} = \frac{\overline{|T|^2}}{16\pi^2 s^2} \qquad (11.11)$$

where $dt = 2E^2\,d\cos\theta$.

Similar results can be found for quark-gluon (QG) and gluon-gluon (GG) scattering. At low momentum transfer, and apart from an overall proportionality constant, their relative squared matrix elements go as

$$\begin{aligned} QQ\to QQ &= \frac{8}{9}\frac{s^2}{t^2} \\ QG\to QG &= 2\frac{s^2}{t^2} \\ GG\to GG &= \frac{9}{2}\frac{s^2}{t^2} \qquad (11.12) \end{aligned}$$

We see that the formulas simplify at small t.

12. GAUGE FIXING THE PATH INTEGRAL* (12-1-87)

We now have a complete set of rules for calculating amplitudes perturbatively in the gauge $A_t = 0$. (It is not perfectly complete because we have not yet specified how to deal with the infinities arising from loop diagrams; this will be the subject of some later chapters.) Ordinarily I would not complicate matters by introducing an additional formalism that gives the same answers in the end, but Faddeev and Popov (Phys. Lett. B25 (1967) pp. 29-30) have invented another way of fixing the gauge that is so elegant and useful that it deserves mention. It allows one to evaluate the path integral with an arbitrary gauge condition, of which $A_t = 0$ is just a special case.

Before deriving the method, it will be useful to know a technique that allows the path integral in $A_t = 0$ gauge,

$$\int e^{iS[\mathbb{A},0]}\,\mathcal{D}^3\mathbb{A}(\mathbb{x},t)\,, \qquad (12.1)$$

to be rewritten as an integral over all four of the A_μ, with an extra A_t-dependent term in the Lagrangian. Notice

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[35] RPF had written $|T|^2/(8\pi^2s^2)$; I have restored the missing factor of 1/2. For same-flavor quarks, there should be yet another factor of 1/2.

that (12.1) is the same as

$$\int e^{iS[\mathbb{A},\phi]}\,\mathcal{D}^3\mathbb{A}\,, \tag{12.2}$$

for any function ϕ. One way to see this is to gauge transform $\mathbb{A}$ in (12.1) to

$$\begin{aligned}\mathbb{A}' &= \Lambda^\dagger\mathbb{A}\Lambda + \Lambda^\dagger\nabla\Lambda \\ A_t' &= \phi = \Lambda^\dagger\dot{\Lambda}\,,\end{aligned} \tag{12.3}$$

where Λ, which is determined by solving $\dot{\Lambda} = \phi\Lambda$, does not depend upon $\mathbb{A}$, hence $\mathcal{D}^3\mathbb{A}$ is invariant. Eq. (12.2) can also be written as

$$\int e^{iS[\mathbb{A},A_t]}\,\mathcal{D}^3\mathbb{A}\,\delta[A_t-\phi] \tag{12.4}$$

Since it does not depend on ϕ, we can functionally integrate over ϕ with some weight, say

$$\exp\left(\frac{i\mu^2}{2g^2}\int d^4x\,\phi^2\right) \tag{12.5}$$

and change the path integral by only an overall multiplicative factor. This factor has no effect on an expectation value of a gauge-invariant functional,

$$\langle F\rangle = \frac{\int\mathcal{D}\phi\,e^{ic\int\phi^2}\int e^{iS[\mathbb{A},A_t]}\,F(\mathbb{A},A_t)\,\mathcal{D}^3\mathbb{A}\,\mathcal{D}A_t\,\delta[A_t-\phi]}{\int\mathcal{D}\phi\,e^{ic\int\phi^2}\int e^{iS[\mathbb{A},A_t]}\,\mathcal{D}^3\mathbb{A}\,\mathcal{D}A_t\,\delta[A_t-\phi]} \tag{12.6}$$

since it cancels between numerator and denominator. Now the ϕ integral is trivial because of the delta functional, and the path integral is

$$\int e^{iS[\mathbb{A},A_t]+\frac{i\mu^2}{2g^2}\int A_t^2\,d^4x}\,\mathcal{D}^4A \tag{12.7}$$

Because of the new term in the action, the gluon propagator now exists, even though A_t is no longer fixed to be zero.

Exercise. Show that the propagator for (12.7) is

$$\begin{aligned}P_{\mu\nu}(k) &= \frac{1}{k^2}\left(-\delta_{\mu\nu}+\frac{k_\mu\eta_\nu}{k\cdot\eta}+\frac{k_\nu\eta_\mu}{k\cdot\eta}-\frac{k_\mu k_\nu\eta^2}{(k\cdot\eta)^2}\right) \\ &- \frac{k_\mu k_\nu\eta^2}{\mu^2(k\cdot\eta)^2}\end{aligned} \tag{12.8}$$

where $\eta_\mu = (1,0,0,0)$.

It would be nice to simplify the propagator by choosing $\mu^2 = -k^2$, which is impossible because μ^2 is just a constant, not a Fourier transform variable. But the same thing can be accomplished by using

$$e^{i\frac{\mu^2}{2g^2}\int d^4x\,(\partial_\mu\phi)^2} \tag{12.9}$$

instead of (12.5) as the weight factor. Then we get (12.8), but with $\mu^2\to -k^2\mu^2$, and μ^2 can be chosen so that the last two terms in (12.8) cancel. This is the justification

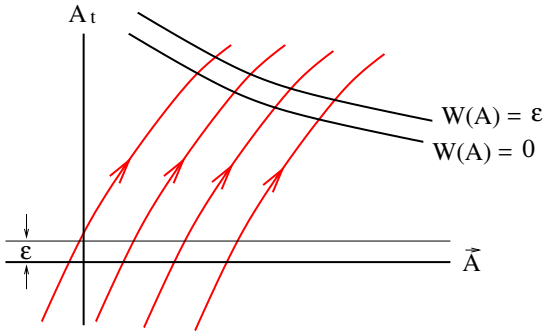


FIG. 28: Gauge orbits sliced by a gauge condition W.

for saying, in a previous lecture, that the $k_\mu k_\nu$ term in the propagator was irrelevant.

We are aiming for an expression similar to (12.7), but for some arbitrary gauge condition, not necessarily $A_t = 0$. For this the Faddeev-Popov procedure will be needed, which since I didn't invent it myself, I claim is extremely subtle! Suppose we wanted the gauge condition to be $\partial_\mu A_\mu = 0$. Then the path integral must look something like

$$\int e^{iS[A]}\,\delta[\partial_\mu A_\mu]\,\mathcal{D}^4A\,. \tag{12.10}$$

But this is not quite right, even though it would be right for the $A_t = 0$ case. Consider the space of all gauge field configurations, represented schematically by plotting A_t along one axis and $\mathbb{A}$ along the other. Let $W(A) = 0$ be the desired gauge condition, represented by a surface in the function space, that cuts across the trajectories of gauge-equivalent $A_\mu(x)$'s, called "gauge orbits" (see fig. 28). Previously we integrated over the line $A_t = 0$. The delta functional $\delta[A_t]$ can be interpreted as the limit of a less singular constraint, which is to integrate over the strip between $A_t = 0$ and $A_t = \epsilon$, divide by ϵ, and take $\epsilon\to 0$. However if we try to do the same thing for the surface $W(A) = 0$, the gauge orbits will not necessarily cross the strip at the same angle everywhere, and the simple constraint $\delta[W(A)]$ will weight some orbits too much, some too little, as one moves along the surface. An extra factor is needed to compensate for the varying length of the orbits crossing the strip.

Call this compensating factor $\Delta(A)$. It will be shown that

$$\Delta(A)^{-1} = \int\mathcal{D}g\,\delta[W(A^g)] \tag{12.11}$$

where

$$A_\mu^g = \Lambda^\dagger(g)\,A_\mu\,\Lambda(g) + \Lambda^\dagger(g)\,i\partial_\mu\Lambda(g)\,, \tag{12.12}$$

is the gauge-transformed A_μ and $\Lambda(g)$ is the matrix representation of the abstract group element g. $\mathcal{D}g$ stands for the invariant group measure at each point in space-

time. It has the property

$$\begin{aligned}\int \mathcal{D}g\,\delta[W(A^g)] &= \int \mathcal{D}g\,\delta[W(A^{hg})] \\ &= \Delta(A^h)^{-1} \end{aligned} \tag{12.13}$$

for any h in SU(3), so we see that $\Delta(A)$ is gauge invariant.

To define the path integral, insert a factor of $1 = \int \Delta(A) \int \mathcal{D}g\,\delta[W(A^g)]$ into $\int e^{iS[A]}\,\mathcal{D}^4A$. The expectation value of $F[A]$ is then

$$\begin{aligned}\langle F\rangle &= \frac{\int e^{iS[A]}\,F\,\mathcal{D}^4A}{\int e^{iS[A]}\,\mathcal{D}^4A} \\ &= \frac{\int e^{iS[A]}\,F\,\Delta(A)\,\delta[W(A^g)]\,\mathcal{D}^4A\,\mathcal{D}g}{\int e^{iS[A]}\,\Delta(A)\,\delta[W(A^g)]\,\mathcal{D}^4A\,\mathcal{D}g}\,. \end{aligned} \tag{12.14}$$

The nice thing about these integrals is that they don't depend on g (assuming, as usual, that F is gauge invariant). Make the change of variables $A \to A^{g^{-1}}$. Because each factor in the integrals is gauge invariant except for the delta functional, the g-dependence disappears, and the (infinite) integrals $\int \mathcal{D}g$ cancel between numerator and denominator. We are therefore left with the path integral

$$Z = \int e^{iS[A]}\,\Delta(A)\,\delta[W(A)]\,\mathcal{D}^4A\,, \tag{12.15}$$

as claimed.

Eq. (12.15) is the desired generalization of (12.1) for $A_t = 0$ gauge, but it is not in a very useful form for explicit computations. $\Delta(A)$ is some horribly complicated functional which in general nobody knows how to compute. Fortunately, it is not necessary to know $\Delta(A)$ for all values of A, but only where $W(A) = 0$, and there it *can* be determined. Take $W(A) = \partial_\mu A_\mu$, for example. We must evaluate

$$\int \delta[\partial_\mu A^g_\mu]\,\mathcal{D}g\Big|_{\partial_\mu A_\mu = 0}\,. \tag{12.16}$$

We first assume that there is a unique solution to $\partial_\mu A^g_\mu = 0$, such that g is the identity when $\partial_\mu A_\mu$ is already zero. Therefore we can focus on infinitesimal gauge transformations,

$$\vec{A}^g_\mu = \vec{A}_\mu + D_\mu\vec{\alpha} = \vec{A}_\mu + (\partial_\mu - A^\times_\mu)\vec{\alpha}\,. \tag{12.17}$$

Then (12.16) becomes

$$\int \delta[\partial_\mu A_\mu + \partial_\mu D_\mu\alpha]\,\mathcal{D}\alpha\Big|_{\partial_\mu A_\mu = 0}\,. \tag{12.18}$$

Recall that for finite-dimensional integrals,

$$\int d^{\,n}x\,\delta^{(n)}(M_{ab}x^b) = \frac{1}{|\det M|}\,.$$

In the present case, eq. (12.18), we get a functional determinant,

$$\mathrm{Det}^{-1}(\partial_\mu D_\mu)\,,$$

which depends on A_μ through D_μ. Therefore the path integral (12.5) is

$$\int e^{iS[A]}\,\delta[\partial_\mu A_\mu]\,\mathrm{Det}(\partial_\mu D_\mu)\,\mathcal{D}A \tag{12.19}$$

in the gauge $\partial_\mu A_\mu = 0$.

The next step is to reexpress the determinant so that it looks like a new term in the action. If instead of $\mathrm{Det}(\partial_\mu D_\mu)$ we had $\mathrm{Det}^{-1/2}(\partial_\mu D_\mu)$, we could use the functional generalization of the formula

$$\int d^{\,n}x\,e^{-\frac{1}{2}x^aM_{ab}x^b} = \frac{(2\pi)^{n/2}}{\sqrt{\det M}}\,.$$

However, there is an analogous formula for anticommuting variables that does what is needed,

$$\int \mathcal{D}P\,\mathcal{D}\bar{P}\,e^{\bar{P}MP} = \det M\,. \tag{12.20}$$

Here P is an anticommuting function, $\{P(x), P(x')\} = 0$, $\bar{P}$ is its complex conjugate, and M is a differential operator. (The reader who is unfamiliar with this type of integral should work through the following exercise.)

Exercise. Complex anticommuting variables are defined to satisfy $\theta^2 = \bar{\theta}^2 = \{\theta, \bar{\theta}\} = 0$. The complete table of integrals for such variables is, by definition,

$$\int d\theta = \int d\theta\,\bar{\theta} = 0; \quad \int d\theta\,\theta = 1$$

(and similarly for the complex conjugates). If there are $2N$ variables θ_i, $\bar{\theta}_i$, $i = 1, \ldots, N$, then they all anticommute with each other. By the above rule, the only nonvanishing integrals over all the θ_i, $\bar{\theta}_i$ are

$$\int d^{\,n}\theta\,d^{\,n}\bar{\theta}\,\left[\bar{\theta}_1\theta_1\ldots\bar{\theta}_N\theta_N\right] = 1\,,$$

and those integrals that differ from it by a permutation of the variables in the integrand. (If the permutation is odd, the integral will be -1 instead of $+1$.) Convince yourself that

$$\int d^{\,n}\theta\,d^{\,n}\bar{\theta}\,e^{\bar{\theta}_aM_{ab}\theta_b} = \det M\,.$$

Applying this technique to the path integral gives

$$\int e^{iS[A]}\,\delta[\partial_\mu A_\mu]\,e^{i\int \bar{P}(x)\partial_\mu D_\mu P(x)\,d^{\,4}x}\,\mathcal{D}P\,\mathcal{D}\bar{P}\,, \tag{12.21}$$

where the P's transform in the octet representation of SU(3), since that is the representation on which D_μ acted in (12.18).

The new fields P, $\bar{P}$ are called Faddeev-Popov *ghosts* to underscore the fact that they are not physical fields like quarks or gluons, but only a mathematical convenience. Because they are anticommuting, they behave like fermions in the sense that ghost loops contribute a

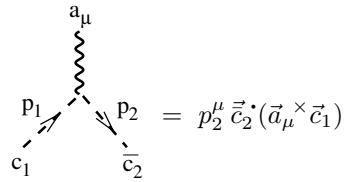
factor of -1 in a Feynman diagram; but they are not fermions, for their spin is zero. The ghost action can be put into a more conventional form by integrating by parts,

$$\partial_\mu \bar{P}\, D_\mu P = \partial_\mu \bar{P}\, \partial_\mu P - \partial_\mu \bar{P} \cdot (A_\mu{}^\times P)\,. \tag{12.22}$$

The Feynman rules for ghosts are seen to be

-1 for loops;

propagator: a - - - - - b $= \dfrac{1}{k^2}\,\delta_{ab}\,;$



coupling : $= p_2^\mu\, \vec{\bar{c}}_2 \cdot (\vec{a}_\mu{}^\times \vec{c}_1)$

where $\vec{c}_1$ and $\vec{\bar{c}}_2$ are octet color vectors, just like $\vec{a}_0, \ldots, \vec{a}_3$. In addition, the gauge field propagator is simply

a,μ ∿∿∿ b,ν $= \dfrac{\delta_{\mu\nu}\,\delta_{ab}}{k^2}$

in this gauge, since

$$\tfrac{1}{4}\int d^4x\, (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 = \tfrac{1}{2}\int d^4x\, \left((\partial_\mu A_\nu)^2 - (\partial_\mu A_\mu)^2\right)$$

after integrating by parts, and $\partial_\mu A_\mu = 0$.

Now we are almost done, but the expression (12.21) is still not easy to use since we don't know how to do Gaussian integrals with a constraint. This is where the trick introduced at the beginning of the chapter comes in. If instead of $\partial_\mu A_\mu = 0$ one used $\partial_\mu A_\mu = f$, $\Delta(A)$ would be the same as before, and (12.21) would be

$$\int e^{iS[A]}\, \Delta(A)\, \delta[\partial_\mu A_\mu - f]\, \mathcal{D}^4 A\,. \tag{12.23}$$

This expression does not really depend on f because it was obtained from the same starting point, $\int e^{iS[A]}\, \mathcal{D}^4 A$, for any f. So again it is permissible to integrate over f with a weight factor $\exp(i\int f^2\, d^4x)$. The final result is

$$\int e^{iS[A] + \frac{i}{2g^2}\int (\partial_\mu A_\mu)^2\, d^4x + i\int \partial_\mu \bar{P} D_\mu P\, d^4x}\, \mathcal{D}^4 A\, \mathcal{D}\bar{P}\, \mathcal{D}P\,. \tag{12.24}$$

The second term in the action cancels the similar term in $S[A]$ so that the gluon propagator is still $\delta_{\mu\nu}/k^2$, but there is no longer any restriction on A_μ in the integral.

[36] In deriving this we assumed that for a given vector field A_μ the gauge transformation g needed to arrange that the divergence of the new field is zero, $\partial_\mu A_\mu^g = 0$, is unique. This was found to be false by Gribov. Thus $\partial_\mu A_\mu = 0$ does not completely specify the gauge. Thus Faddeev's argument looks imperfect—first there are several places in g where there are contributions to (12.15). In addition our ghost gives $\mathrm{Det}(\partial_\mu D_\mu)$ but our analysis from (12.18) wants the absolute value $|\mathrm{Det}(\partial_\mu D_\mu)|$. There is much confusion, but I think (from some studies I made some time ago) the final integral is really correct.[37] At any rate no error would be expected in perturbation theory because for configurations with small A_μ the gauge that makes $\partial_\mu A_\mu = 0$ is unique. Gribov's ambiguity appears only for sufficiently large A. We shall see examples of it later.

Exercise. Rewrite (12.24) for a general gauge condition, $W(A)$.

13. QUARK CONFINEMENT* (12-3-87)

The utility of being able to quantize QCD in a variety of different gauges is that some gauges are particularly convenient for certain applications. In electrodynamics, the Coulomb gauge has the virtue that fields satisfying the gauge condition

$$\mathbb{\nabla} \cdot \mathbb{A} = 0 \tag{13.1}$$

represent truly physical, transverse degrees of freedom. It also has some peculiarities. The Hamiltonian has a nonlocal, instantaneous interaction (action at a distance), which must combine with the interactions of the transverse photons so that the net force propagates at the speed of light.

Interactions that look instantaneous are well suited to Schrödinger's equation, which requires the potential between particles at equal times. It would be quite awkward to explicitly describe finite-velocity forces in the Schrödinger equation because the potential for one particle at a time t would depend on the positions of the others at the retarded times, and one would need the past histories of all the particles to propagate the system forward in time. In what follows we will derive the Coulomb gauge path integral for QCD in the Hamiltonian form (which is closely related to the Schrödinger picture) and see some indications of quark confinement.

Write the gauge field as

$$A_\mu^i = (\phi^i,\, \mathbb{A}^i)\,. \tag{13.2}$$

Then in the gauge (13.1), the vacuum amplitude is

$$X = \int e^{iS[A]}\, \mathrm{Det}(\mathbb{\nabla} \cdot \mathbb{D})\, \delta[\mathbb{\nabla} \cdot \mathbb{A}]\, \mathcal{D}^3\mathbb{A}\, \mathcal{D}\phi\,, \tag{13.3}$$

[36] The following paragraph was added to my revision of the lecture by RPF.

[37] In my course notes, I have some elaboration of this point: "RPF conjectures that the sign of the Det changes only if the gauge condition does not uniquely determine A_μ. In this case, he believes that integrating over the different solutions gives a compensating error that makes up for using Det instead of |Det|."

where $\mathbb{D} = (\mathbb{\nabla} - \mathbb{A}^{\times})$ is the covariant derivative in the adjoint representation. The constraint in (13.3) implies that if $\mathbb{A}$ is expanded in plane waves, $\mathbb{C}(\mathbb{k})e^{i\mathbb{k}\cdot\mathbb{x}}$, then $\mathbb{k}\cdot\mathbb{C} = 0$. The color electric and magnetic fields are (omitting the gauge index for brevity)[38]

$$\begin{aligned} \mathbb{E} &= (-\mathbb{\nabla}\phi + \dot{\mathbb{A}} + \mathbb{A}^{\times}\phi) = \dot{\mathbb{A}} - \mathbb{D}\phi \qquad (13.4)\\ \mathbb{B} &= \mathbb{\nabla}_{\times}\mathbb{A} - \tfrac{1}{2}\,\mathbb{A}^{\times}_{\times}\mathbb{A} \qquad (13.5)\end{aligned}$$

and the action is

$$S[A] = \frac{1}{2g^2}\int (\mathbb{E}^2 - \mathbb{B}^2)\, d^4x + \int (\rho\phi - \mathbb{J}\cdot\mathbb{A})\, d^4x \quad (13.6)$$

in the presence of an external source $J^{\mu} = (\rho, \mathbb{J})$. Notice that $\exp(\frac{i}{2g^2}\int \mathbb{E}^2)$ can be rewritten as

$$e^{\frac{i}{2g^2}\int \mathbb{E}^2\, d^4x} = c\int e^{-\frac{ig^2}{2}\int \mathbb{\Pi}^2\, d^4x + i\int \mathbb{\Pi}\cdot\mathbb{E}\, d^4x}\, \mathcal{D}^3\mathbb{\Pi} \quad (13.7)$$

by completing the square. Then the amplitude becomes

$$\begin{aligned} X &= \int e^{-\frac{ig^2}{2}\int \mathbb{\Pi}^2\, d^4x + i\int \mathbb{\Pi}\cdot(\dot{\mathbb{A}} - \mathbb{D}\phi)\, d^4x + \frac{i}{2g^2}\int \mathbb{B}^2\, d^4x}\\ &\times\ e^{i\int(\rho\phi - \mathbb{J}\cdot\mathbb{A})\, d^4x}\, \mathrm{Det}(\mathbb{\nabla}\cdot\mathbb{D})\, \mathcal{D}^3\mathbb{A}_T\, \mathcal{D}\phi\, \mathcal{D}^3\mathbb{\Pi} \end{aligned} \quad (13.8)$$

where $\delta[\mathbb{\nabla}\cdot\mathbb{A}]$ has been eliminated by integrating only over the transverse part of $\mathbb{A}$, denoted by $\mathbb{A}_T$, and the integrand is evaluated at $\mathbb{\nabla}\cdot\mathbb{A} = 0$.

Now any vector field can be split into a longitudinal and a transverse part,

$$\begin{aligned} \mathbb{\Pi} &= \mathbb{\Pi}_{\text{transverse}} + \mathbb{\Pi}_{\text{longitudinal}}\\ &\equiv \mathbb{P} + \mathbb{\nabla} f \end{aligned} \quad (13.9)$$

where $\mathbb{\nabla}\cdot\mathbb{P} = 0$, by definition. Then

$$\begin{aligned} X &= \int \exp\Bigg[-\frac{ig^2}{2}\int \left(\mathbb{P}\cdot\mathbb{P} + \underline{2\mathbb{P}\cdot\mathbb{\nabla} f} + (\mathbb{\nabla} f)^2\right) d^4x\\ &+\ i\int \mathbb{P}\cdot\left(\dot{\mathbb{A}} - \underline{\mathbb{\nabla}\phi} + \mathbb{A}^{\times}\phi\right) d^4x + i\int (\rho\phi - \mathbb{J}\cdot\mathbb{A})\, d^4x\\ &+\ i\int \mathbb{\nabla} f\cdot(\underline{\dot{\mathbb{A}}} - \mathbb{D}\phi)\, d^4x + \frac{i}{2g^2}\int \mathbb{B}\cdot\mathbb{B}\ d^4x\Bigg]\\ &\times\ \mathrm{Det}(\mathbb{\nabla}\cdot\mathbb{D})\, \mathcal{D}\mathbb{A}_T\, \mathcal{D}\phi\, \mathcal{D}\mathbb{P}\, \mathcal{D}f\,. \end{aligned} \quad (13.10)$$

The underlined terms can be eliminated by integrating by parts and using the fact that $\mathbb{\nabla}\cdot\mathbb{P} = \mathbb{\nabla}\cdot\mathbb{A} = 0$. An overall constant, $\mathrm{Det}(\mathbb{\nabla})$, has been omitted from the functional measure $\mathcal{D}f$. Grouping the remaining ϕ-dependent factors together and integrating over ϕ gives a delta functional,

$$\delta[\mathbb{P}^{\times}\mathbb{A} + \mathbb{D}\cdot\mathbb{\nabla} f + \rho] \quad (13.11)$$

which says that f must satisfy

$$f = -(\mathbb{D}\cdot\mathbb{\nabla})^{-1}(\rho + \mathbb{P}^{\times}\mathbb{A})\,. \quad (13.12)$$

This is reminiscent of the analogous equation in electrodynamics,

$$\tilde{f} = -\frac{1}{\nabla^2}\rho\,,$$

whose solution is

$$\tilde{f} = \frac{1}{4\pi}\int \frac{\rho(\mathbb{R}')}{|\mathbb{R} - \mathbb{R}'|}\, d^3\mathbb{R}'\,, \quad (13.13)$$

but the complexity of the QCD version (13.12) prevents us from obtaining such a nice closed-form solution for f.

There is a fortunate simplification from the constraint (13.11) however. The delta functional produces a factor of

$$\mathrm{Det}^{-1}(\mathbb{D}\cdot\mathbb{\nabla}) \quad (13.14)$$

when the integral over f is performed (recall that $\delta(ax) = \delta(x)/a$). If it was $\mathrm{Det}^{-1}(\mathbb{\nabla}\cdot\mathbb{D})$ instead, it would cancel the Faddeev-Popov determinant in (13.10). However, $\mathbb{\nabla}\cdot\mathbb{D}$ is the same as $\mathbb{D}\cdot\mathbb{\nabla}$ in the present case, since

$$\mathbb{D}\cdot\mathbb{\nabla} g = (\mathbb{\nabla} - \mathbb{A}^{\times})\cdot\mathbb{\nabla} g = \nabla^2 g - \mathbb{A}^{\times}_{\cdot}\mathbb{\nabla} g$$

and

$$\mathbb{\nabla}\cdot\mathbb{D} = \mathbb{\nabla}\cdot(\mathbb{\nabla} - \mathbb{A}^{\times})g = \nabla^2 g - (\mathbb{\nabla}\cdot\mathbb{A})^{\times} g - \mathbb{A}^{\times}_{\cdot}\mathbb{\nabla} g\,;$$

the $\mathbb{\nabla}\cdot\mathbb{A}$ term vanishes in the path integral (13.10) which can therefore be written as

$$\begin{aligned} X &= \int e^{i\int \mathbb{P}\cdot\dot{\mathbb{A}} - \mathcal{H}(\mathbb{P},\mathbb{A})]\, d^4x}\, \mathcal{D}\mathbb{A}_T\, \mathcal{D}\mathbb{P}\,; \qquad (13.15)\\ \mathcal{H} &= \frac{1}{2g^2}\left((\nabla f)^2 + \mathbb{P}\cdot\mathbb{P} + \mathbb{B}\cdot\mathbb{B} + \mathbb{J}\cdot\mathbb{A}\right) \qquad (13.16)\end{aligned}$$

This is the Hamiltonian form of the path integral. Let us recall from ordinary quantum mechanics the connection between it and the Lagrangian form,

$$X_L = \int e^{iS}\mathcal{D}Q(t) \quad (13.17)$$

where Q is the particle coordinate, $S = \int L dt$, and the Lagrangian is

$$L = \frac{m}{2}\dot{Q}^2 - V(Q) \quad (13.18)$$

for a particle of mass m. From L one derives the canonical momentum

$$P = \frac{\partial L}{\partial \dot{Q}} = m\dot{Q} \quad (13.19)$$

and Hamiltonian

$$H = (P\dot{Q} - L)\Big|_{\dot{Q} = P/m} = \frac{P^2}{2m} + V(Q)\,. \quad (13.20)$$

[38] See note [25].

The Hamiltonian path integral is given by

$$\begin{aligned} X_H &= \int e^{i\int[P(t)\dot{Q}(t)-H(P,Q)]dt}\,\mathcal{D}P(t)\,\mathcal{D}Q(t) \\ &= \int e^{i\int[P(t)\dot{Q}(t)-\frac{P^2}{2m}-V(Q)]dt}\,\mathcal{D}P\,\mathcal{D}Q \end{aligned} \tag{13.21}$$

This is seen to be the same as X_L after completing the square and integrating over P. However, it would do no good to carry out the integral over $\mathbb{P}$ in the case of QCD, (13.16), because it would introduce a complicated functional determinant depending on $\mathbb{A}_T$, due to the way $\mathbb{P}$ enters $\mathbb{V}f$, (12.12).

We now concentrate on the interaction Hamiltonian from (13.16),

$$\begin{aligned} \mathcal{H}_I &= \frac{1}{2g^2}\int(\mathbb{V}f)^2 d^3x = -\frac{1}{2g^2}\int f\nabla^2 f\, d^3x \\ &= -\frac{1}{2g^2}\int\left[\rho+\mathbb{P}\overset{\times}{\cdot}\mathbb{A}\right]\frac{1}{\mathbb{D}\cdot\mathbb{V}}\nabla^2\frac{1}{\mathbb{D}\cdot\mathbb{V}}\left[\rho+\mathbb{P}\overset{\times}{\cdot}\mathbb{A}\right]d^3x \end{aligned} \tag{13.22}$$

You will recall that ρ is the charge density of the external quark field. This suggests interpreting $\mathbb{P}\overset{\times}{\cdot}\mathbb{A}$ as the color charge density of the transverse gluons. To see that this interpretation makes sense, recall that a complex scalar field $\phi=(\phi_1+i\phi_2)$ has the charge density

$$\begin{aligned} \rho &= \frac{1}{2i}\left(\phi^*\dot{\phi}-\dot{\phi}^*\phi\right) = \frac{1}{2i}\left(\phi^*\pi^*-\pi\phi\right) \\ &= (\phi_1\pi_2-\phi_2\pi_1) \end{aligned} \tag{13.23}$$

where $\pi=\partial\mathcal{L}/\partial\dot{\phi}$ is the canonically conjugate momentum. Eq. (13.23) is like the cross product of the vectors (ϕ_1,ϕ_2) and (π_1,π_2).

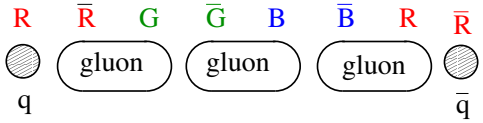
Now we would like to deduce the potential between quarks from the operator $(\mathbb{D}\cdot\mathbb{V})^{-1}\nabla^2(\mathbb{D}\cdot\mathbb{V})^{-1}$ in (13.22). Although $\mathbb{D}\cdot\mathbb{V}$ cannot be inverted in closed form, it can be expanded in powers of the gluon field, which corresponds to weak coupling. We get

$$\begin{aligned} \frac{1}{\mathbb{D}\cdot\mathbb{V}}\nabla^2\frac{1}{\mathbb{D}\cdot\mathbb{V}} &= \frac{1}{\nabla^2}+2\,\frac{1}{\nabla^2}\mathbb{A}\overset{\times}{\cdot}\mathbb{V}\frac{1}{\nabla^2} \\ &+ 3\,\frac{1}{\nabla^2}\mathbb{A}\overset{\times}{\cdot}\mathbb{V}\frac{1}{\nabla^2}\mathbb{A}\overset{\times}{\cdot}\mathbb{V}\frac{1}{\nabla^2} \\ &+ \ldots \end{aligned} \tag{13.24}$$

The first term corresponds to the Coulomb potential, as in (13.13).

[39] If the following terms are evaluated to 2nd order we get a strong attraction. In $\beta_0=11-\frac{2}{3}n_f$[40] it contributes $+12$ units—most of the confining effect. Thus the instantaneous Coulomb interaction probably rises with distance. The charge densities of transverse ("real") gluons and quarks are $(\rho+\mathbb{P}\overset{\times}{\cdot}\mathbb{A})$ and their interaction via $\mathcal{H}_I$, via $(\mathbb{D}\cdot\mathbb{V})^{-1}\cong\nabla^{-2}$, to second order in $\mathcal{H}_I$, makes a vacuum polarization of the normal sign, contributing $-1-\frac{2}{3}n_f$ to β_0.

This leads to an interesting model of a string connecting heavy quarks. Because of the rapid rise (with r^2)[41] of the force between charges, unbalanced color charges $\rho+\mathbb{P}\overset{\times}{\cdot}\mathbb{A}$ at any distance are intolerable. Suppose we start with a red quark at $r=0$ and say anti-red far away to the right. By creating a dipole gluon $\bar{R}G$ within this range where opposite colors are tolerable we cancel redness at larger transverse distances. But then the G end of the gluon is unbalanced, so another gluon dipole $\bar{G}B$ forms (the energy for which coming from the decrease in G energy). This continues until we meet the final quark. Thus we have a state of superpositions of color arrangements,



In the transverse region away from the string, all the colors cancel and there is not much energy in the Coulomb-like quantum chromostatic field.

We shall have a more general and precise discussion of these strings later on in the course, and will see if this is a useful viewpoint of a string. (Such pictures have also been discussed by Greenstreet.[42])

14. INTERLUDE (1-5-88)

[From the audio tape. RPF begins with some remarks about the research project.] *First, last term we added an assignment to write something up; I've given some of them back already and here are the rest. I've written a lot on them but it doesn't mean that I've corrected everything. Mainly I'm trying to suggest some ways of looking at things, which is not meant as a criticism necessarily; well it could be a criticism, but very often a great deal of what I'm writing is not a criticism. I think this is a very useful way of teaching, and I would like to do the same thing this term, at the end of this term, having another paper to hand in on the first day of exam week at the end of the term.*

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[39] The remainder of this section was added to my submitted draft by RPF.

[40] The beta function has not yet been introduced; this will come in the next lecture.

[41] This seems to be a slip since the equations imply a linear potential between charges, leading to a constant force, as is usually understood. It can be derived using dimensional analysis, by evaluating the Green's function ∇^{-2} at large separations.

[42] RPF means Greenberg, who wrote several papers about quark confinement.

And this paper this term will have to be better than the paper of the last term. You remember that I decided on that rather late; I only gave you a few weeks. So I said it doesn't have to be too good. So everybody passed. On some of the papers I was rather surprised and disappointed and thought that you were unable or didn't have the time to do your subject justice, and I made some remarks that you should talk to me about it or something like that; that's only one or two so don't worry. Sometimes a person isn't prepared or doesn't have the background or the focus needed, and it's worth finding out earlier rather than later.

The other possibility is that you've got some kind of a block somewhere, a misunderstanding of what it's all about. And it's very surprising [noise] *... it looks very simple to me, but it's only a block. I know that because I've had that experience once myself, and I understand it. I also used to do tutoring, and I've discovered many blocks like this ... and straightened the person out who had the wrong idea. I'll give you an example. My own block occurred when I studied solid geometry in high schoool. I was pretty good at math as you might guess, and I thought I'd have a good time in solid geometry. The class started and I didn't understand anything. The guy would ask questions, and I couldn't figure out what the hell the answer was, and moreover something that I thought was usually pretty dopey would come out as the right answer. It would proceed like this and I was getting foggier and foggier for about a week. All of a sudden,* **thank God***, it suddenly hit me. I understood. They were drawing things like this, parallelograms, overlapping. Then sometimes lines that would come out* [connecting the parallelograms]*; the question was were they perpendicular to another line like that ...* [laughter]*. I was looking at every one of the diagrams as though it were a plane diagram, but it was a three-dimensional picture. So when I finally figure it out the teacher told me "that's why we call it solid geometry, you idiot!"* [laughter]*. I couldn't figure out the theorems and equations and relations because I was looking at the diagrams flat. That's a block.*

Another example was a student that a lady came to me and told me that her son was very good in math but was needing some help in geometry, he was in high school. So the first thing I did was to ask him some questions, like if this is a rectangle and this is the diagonal, how far is it from here to here? It's the same, he says; and then a few more questions, showing that he had a first-rate intuition about geometry; there was no problem about understanding what geometry was about. But to make a long story short, he had a double block. One: they had in his course in high school that you write what they call proofs. You write something and then you write the reason. Given: ... and so on and so on. The way he thought was, and he didn't understand, was how you knew what to write on each line. He thought there was some logical process to know what to write on each line. And I had to explain to him, no, what you did was you first had to figure out how to prove it. And then after you figured out how to prove it you wrote the proof down. Okay, that was stupid, but that's the way they did it.

Another block was, he didn't realize the rule that the theorems that you were allowed to use in proving, the things you could write on the right-hand side as reasons, always had to be statements that were earlier than the thing you were trying to prove. For example, no, you can't use the Pythagorean theorem here because you haven't gotten there yet. Sounds dumb, but it's just artificial conventions of human beings [long section where RPF is speaking away from the microphone].

Same thing happens in algebra, where people have great difficulty because it's not realized and it's not explained to them that x is used in two ways to represent a number; when you have a problem like this [writing on board]*. The problem is x is some special number and you have to find it, and when you have a statement like this* [more writing]*that's an entirely different use; there it means it's true no matter what x is.*

So these blocks may be the cause of some ... if you look at your papers and find some remark ... let's talk about it ... a chance to maybe figure out if you have some difficulty. Alright? Most of you have no such difficulty.

Is everything alright? Do you want me to give more problems during the year, or is it okay if we do these final exam papers, because they're very useful, I think ... If you have some objections, come and tell me because I know it's been ...

15. SCALE DEPENDENCE (1-5-88)

So far in this course we have emphasized perturbation theory as the main tool for making theoretical predictions, but it is important to keep in mind that the path integral is not limited to this treatment. For example, lattice gauge theory, which we will discuss in some detail, is a way of computing observables nonperturbatively. And I will bring up some other possible ideas for going beyond perturbation theory. That will occupy us in the second part of this term. For the first part, we will continue to explore aspects of perturbation theory.

I would like to spend some time in this lecture on the topic of running coupling constants, which must be handled carefully in order to avoid confusion. In fact, there is quite a great potential for confusion in this subject, and an apparent complexity, because of the lack of agreement about the best conventions for carrying out the renormalization of the couplings, and also some misguided suggestions that the running coupling should be defined in terms of some particular processes.

15.1. Measuring couplings

But before we discuss the running, we should try to understand what are the most efficient ways of experimentally determining the values of the couplings in the

QCD Lagrangian,

$$\frac{1}{g_0^2}\mathrm{tr}FF + \sum_i \bar{q}_i(i\partial\!\!\!/ + m_0^i)q_i \tag{15.1}$$

where I have summed over the quark flavors, $i = u, d, s, c, b$ (and presumably t, although it has not yet been observed). The subscript 0 means that these are the bare couplings, that would coincide with the physical values if we were to make predictions only at tree level, but which of course will differ once we start to include loops. So we have at least six Lagrangian parameters,

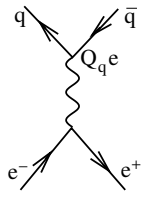
$$g_0^2,\ m_0^u,\ m_0^d,\ m_0^s,\ m_0^c,\ m_0^b \tag{15.2}$$

In principle, we could determine all of these by measuring six independent observables, since generically each one would constrain different combinations of the parameters. But in practice we usually focus on one thing at a time, and try to choose an observable that is most sensitive to the quantity of interest. For example, to compute m_0^b, we could initially estimate it as approximately half the mass of the Υ meson (bound state of $b\bar{b}$), since we know that Υ gets most of its mass from the quarks and not the gluons. This would give $m_b^0 \sim 5\,\mathrm{GeV}$. Similarly we can estimate m_c^0 as being about half of the mass of the ψ. Of course the Υ and ψ do get some of their mass from the gluons, and so we could try to improve on these estimates by doing a bound-state calculation to take that into account—but that would depend on other parameters, namely g_0^2.

To determine g_0, we could try to compute the mass of a particle like the proton, which is believed to get most of its mass from the gluons, since $m_0^u,\ m_0^d \ll m_p$. However we then encounter the problem that we don't know how to compute m_p; it is very far from being a perturbative calculation, and we would have to rely on the lattice, which for the present looks hopeless, although maybe someday it will be feasible.

Instead, to measure g_0 it is more practical to look at a process involving higher energies, much higher than the quark masses, so that the measurable quantity depends very weakly on the m_0^i; in this way we can disentangle the dependences and determine g_0 independently. At arbitrarily high energy $E \gg m_0^i$ this should become an increasingly good approximation since the amplitudes will depend only on the ratio m_0^i/E.

Let me start with a process that at first looks like it will not help us, since it does not seem to depend on the QCD coupling g_0 at all: electroproduction of quarks:



(15.3)

Of course the real physical process does depend on g_0 because what we actually observe is not the quarks, but rather $e^+e^- \to$ hadrons, and the hadronization process depends strongly on g_0. But this is not the kind of g_0 dependence we are interested in because (as I will argue later in the course) it is more characteristic of the low-energy scales of the hadron masses and it is not a perturbatively computable process. Instead, we want to ignore the details of hadronization and pretend that the quarks are produced freely. This is actually a much better approximation than one might at first imagine, for the simple reason that hadronization mainly occurs *after* the quarks have been produced. Therefore it cannot affect the production cross section (at least at energies that are not too close to resonances), which is the quantity that we *can* compute perturbatively, and which we can measure more or less cleanly despite the complications of hadronization.

Hence the electroproduction process does not depend on g_0 at all, at the leading order in couplings; I am going to take advantage of a subleading effect to suggest a way of measuring g_0. But first let's look at the leading contribution, in the high-energy limit where we can ignore all the masses. We see that the calculation for producing quarks is hardly different from that for producing $\mu^+\mu^-$. The only differences are the charges of the quarks, and the fact that quarks come in three colors. Consider $e^+e^- \to u\bar{u}$; the charge of u is $+2/3$, so we can relate the quark and lepton cross sections as

$$\sigma_{u\bar{u}} = 3\left(\frac{2}{3}\right)^2 \sigma_{\mu^+\mu^-}$$

The factor of 3 is commonly understood as coming from the number of quark colors: you can produce either a red quark, or a blue or a green. This is actually a cheat: in reality you can only produce a color-singlet combination

$$\frac{1}{\sqrt{3}}\left(R\bar{R} + B\bar{B} + G\bar{G}\right) \equiv \frac{3\,a}{\sqrt{3}} = \sqrt{3}\,a \tag{15.4}$$

In this formula, a represents the amplitude for producing a quark of a definite color, that we assumed could occur when we multiplied its cross section by 3. We see that the amplitude of the color singlet state that is actually produced is bigger by a factor of $\sqrt{3}$, and so this is the proper explanation of the factor of 3 in the cross section. But either way of thinking about it gives the right answer.

Similarly for producing $d\bar{d}$, we have

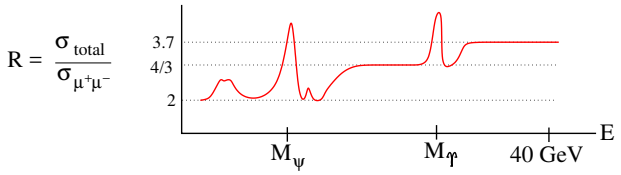
$$\sigma_{d\bar{d}} = 3\left(-\frac{1}{3}\right)^2 \sigma_{\mu^+\mu^-},$$

and we can continue this to include all the higher mass quarks if the center-of-mass energy is sufficient to produce them.

The interesting quantity to measure and compare to predictions is the ratio

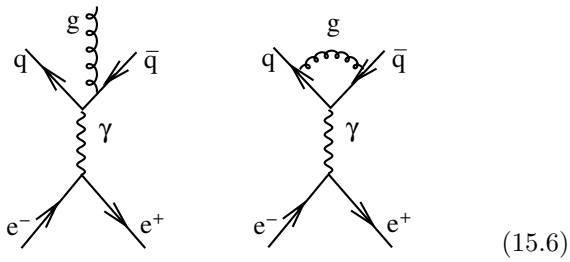
$$R = \frac{\sigma_{\mathrm{total}}}{\sigma_{\mu^+\mu^-}} \tag{15.5}$$

whose contributions, at high enough Q^2, we can read off from equations like (15.4,15.5). The experimental data look like this:



At each quark threshold, a resonance for the corresponding bound state occurs, giving rise to the peaks and dips. These are the hadronization complications that we would like to avoid. In between the thresholds, we observe the simple behavior that we can understand within our approximations: R is just flat, and it is counting the number of quarks that can be produced, weighted by their electric charges (squared).

Now this is all very beautiful, but it doesn't yet help us to determine the QCD coupling. For that, we should consider a higher-order process, where a gluon gets radiated from one of the quarks, or exchanged:



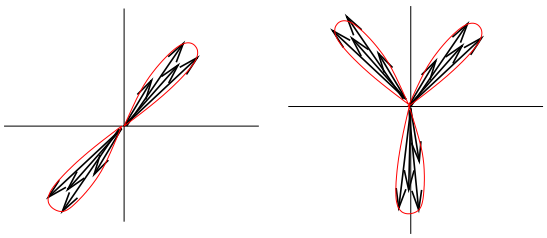
(15.6)

One can show that at high energies, these processes are also independent of the quark masses, and they lead to a multiplicative correction of the leading order prediction R_0,

$$R = R_0 \left(1 + \frac{\alpha_g}{\pi} + \dots\right) \tag{15.7}$$

where $\alpha_g = g^2/4\pi$. Then if we can measure R well enough, the deviation between R and the lowest order prediction R_0 will determine α_g. So this is a possible way of measuring g_0. It is not quite as direct as we would like, since it depends on a relatively small correction, of order a few percent, to the basic quantity that is insensitive to g_0. Can we do better?

A more direct approach would be to observe the gluon that is emitted. That of couse is impossible, just like for the quarks, since they are all colored objects. Instead, we observe the jets of hadronized particles emerging from those primary particles,



(15.8)

where I have drawn the momentum vectors of the individual hadrons and enclosed them with an envelope to indicate the jet. At high energies, the jets are well-defined, with lengths (in momentum space) of order the center-of-mass energy E, and widths of order the QCD scale $\sim 1\,\mathrm{GeV}$. The two-jet event corresponds to the leading order process (15.3), while the three-jet event comes from the first diagram in (15.6), assuming the gluon was radiated at a large enough angle to produce a distinct jet.

As we will discuss in a later lecture, it could happen that the gluon is too soft, is radiated nearly in the same direction as the quark, and its hadronization products get lumped in with the quark jet. But at high enough energies, there is a significant probability for the gluon jet to be well separated from that of the quarks. It will never be as probable as the soft gluon emission, because the extra quark propagator in the radiative diagram would like to be on shell, which favors the soft emission, but at high energies there is the competing effect from the available phase space to make the hard gluon jet observable.

Of course there is no definite dividing line between hard and soft, so this is somewhat less clean in practice than the idealized pictures shown in (15.8); we might confuse some of the hadrons that came from the gluon as being associated with the quark jet, or vice versa, leading to errors in the estimates of the total momenta of the respective jets. These errors become relatively smaller at high energies, where the soft jet contributions become relatively less important. The important point is, this is an observable that depends directly on α_g, so it is a more senistive determination than using the ratio R. In the following discussion, I will continue to focus on high-energy processes that allow us to neglect the quark masses and focus on the coupling g_0.

15.2. Ultraviolet divergences

You may have noticed something peculiar about the diagrams I drew in (15.6): they mix up different orders of perturbation theory. And of course there are other loop diagrams not shown, like the self-energy correction to the quark from a gluon loop. As you probably know from a previous course on electrodynamics, the tree diagram with the gluon emission *cannot* properly be separated from this self-energy correction: the infrared divergence of the soft gluon emission is canceled by a similar divergence in the self-energy diagram, when it interferes with the tree diagram. So that is one reason we cannot avoid considering the loops. But a more important one, for the present discussion, is that the loops will modify our predictions at high energies, which I have stressed is the best regime for comparing predictions to experiment.

And as you know, these loop contributions are problematic because they diverge at high virtual momenta, so we need to introduce the procedure for cutting off the divergences. A typical kind of integral that we have to

deal with (continuing in our approximation of neglecting masses) is

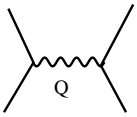
$$\int \frac{d^4k}{k^2(p-k)^2}\,, \tag{15.9}$$

which by counting powers of momenta in the numerator and denominator is logarithmically divergent. A naive way of regularizing this, which later we will see does not quite work in the case of gauge theories, is to modify the propagators by taking

$$\frac{1}{p^2} \to \lim_{\Lambda\to\infty} \frac{1}{p^2} - \frac{1}{p^2-\Lambda^2} \tag{15.10}$$

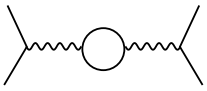
This will render (15.9) finite, in the intermediate step before taking the limit $\Lambda \to \infty$, and yields a divergent term going as $\ln\Lambda$. This divergence can be absorbed by redefining the coupling g_0, before taking the limit.

Let's recall how this works in electrodynamics, for electron-electron scattering. First consider the low-energy, large distance limit, corresponding to scattering in the Coulomb potential e^2/r. The tree-level scattering amplitude is



$$\sim \frac{4\pi e^2}{Q^2} \tag{15.11}$$

I want to consider the $Q \to 0$ limit first, since in this limit the charge e becomes the familiar constant value $e_{\rm exp} \cong \sqrt{1/137} \cong 0.09$.[43] This is the physical value, which is not the same as the bare value e_0 that is needed to cancel the divergences from loops, such as



(15.12)

One can show that the divergences from this loop (plus the others not shown) can be canceled by defining e_0 such that

$$\frac{1}{e^2_{\rm exp}} = \frac{1}{e_0^2(\Lambda)} + \frac{1}{3\pi}\ln\frac{\Lambda^2}{Q^2} \tag{15.13}$$

When we originally wrote the Lagrangian for QED, we thought that e_0 was a parameter very close in value to the measured charge; now we see that it is not a fixed parameter, but rather a function of the cutoff. In the end nothing can depend on Λ, since we are taking the limit $\Lambda \to \infty$. What about the dependence on Q^2? We thought that $e^2_{\rm exp}$ is supposed to go to 0.09 as $Q^2 \to 0$. In reality, we should have something like $Q^2 + m_e^2$ in the log; then this makes sense as $Q^2 \to 0$. The expression (15.13) is valid for $Q^2 \gg m_e^2$.

Of course there is one peculiarity in QED: we *can't* take the $\Lambda \to \infty$ limit! There is some scale $\Lambda \sim m_e\, e^{3\pi\times 137/2}$ at which e_0 diverges, and we cannot make sense out of the theory beyond this point since e_0^2 becomes negative. This is the Landau pole. In practice, it is such a high scale, far greater than the mass of the universe, that we don't care: the residual dependence of amplitudes on Λ is negligible. It is more of an aesthetic shortcoming. But a very nice feature of QCD is that the analogous correction to $1/g_0^2$ has the opposite sign, so there is never any Landau pole; instead g_0 goes to zero as $\Lambda \to \infty$.

In this course I will discuss three different methods of regularization, which are summarized as follows:

1. Pauli-Villars: $\quad \dfrac{1}{k^2} \to \dfrac{1}{k^2} - \dfrac{1}{k^2-\Lambda^2}$

As I mentioned, this is not quite right as written; we need to be more careful to avoid spoiling gauge invariance; we will come to that later.

2. Dimensional regularization:
   α_g becomes dimensionful when $d \neq 4$;
   $\alpha_g \to \alpha_g\, \mu^{d-4}$

where we analytically continue amplitudes in the number d of spacetime dimensions, and

3. Lattice regularization:
   has a dimensionful parameter:
   $a \leftrightarrow 1/\Lambda$

where we have approximated spacetime as a discrete lattice, with sites separated by the lattice spacing a, that plays the role of $1/\Lambda$ relative to the first method. Any of these approaches leads to equivalent results when we take the continuum limits, $\Lambda \to \infty$, $\epsilon \to 0$, $a \to 0$. Each one of them has its own different expression for how the bare couplings depend on the cutoff, but the physical predictions that they make to a given order in perturbation theory are the same. To fully define the theory we will also have to discuss gauge fixing.

Let me continue to work in the Λ cutoff scheme for now; we will come back to the other regulator methods in later lectures. One can show that there is a general form for the Λ dependence of the bare coupling, which we will derive later on,

$$\begin{aligned}\frac{1}{g_0^2(\Lambda)} \;&=\; \beta_0\,\ln\frac{\Lambda^2}{M^2} + \frac{\beta_1}{\beta_0}\ln\left(\ln\frac{\Lambda^2}{M^2}\right) \\ &+\; c_0 + \left(\frac{c_1}{\ln\Lambda^2/M^2} + \ldots\right).\end{aligned} \tag{15.14}$$

Here $\cdots$ denote terms falling with Λ even faster than $1/\ln\Lambda^2$, that do not concern us in the large-Λ limit. The

[43] Notice RPF's unconventional normalization of the coupling, that we have seen before and which he sometimes abandons in favor of the usual one in later lectures.

interesting observation is that the constants β_0 and β_1 are the same in different regularization methods, if we make the appropriate identifications $\ln\Lambda^2 \to 1/\epsilon$ or $\ln\Lambda^2 \to \ln(1/a^2)$ to translate between them. There is an arbitrary scale M appearing, known as the renormalization scale. Notice that any change $M^2 \to e^{-\gamma}M^2$ can be absorbed into a change of the constants c_0 and c_1 (for $\Lambda \gg M$):

$$\frac{1}{g_0^2} \;\to\; \frac{1}{g_0^2} + \underbrace{\beta_0\gamma}_{\delta c_0} + \underbrace{\frac{\beta_1\,\gamma}{\beta_0}}_{\delta c_1} \ln\frac{\Lambda^2}{M^2} + O\left(\ln^{-2}\frac{\Lambda^2}{M^2}\right) \tag{15.15}$$

Thus the c_i are not universal at all, and they also depend on the method of regularization. Their values get fixed, within the given method, by comparing some prediction in which g_0^2 appears to its corresponding measured value. The essential fact is that it should not matter which observable we choose; any quantity that is sufficiently sensitive to g_0 will suffice. And once that is done, we can use the same formula to predict other observables, and use these to test the theory.

You may wonder why I bother to write the c_1 term at all, since it is irrelevant as we take the continuum limit. There may be some practical situations where we are not able to take this limit, notably the lattice, where it is computationally prohibitive to do so. The best we can do there is to take a to some small value. To get the best description of the data at two different values of a, we would need to keep the c_1 term, so as to properly compare predictions at different values of a, to see whether we think a is small enough to trust our predictions.

It is worth remarking on the observation that for QCD, $g_0 \to 0$ as we remove the cutoff. Does that mean that perturbation theory should always work in QCD? No, it depends on the scale of the physical problem. This is because the limit $\Lambda \to \infty$ is the same as the limit $M \to 0$ in eq. (15.14). When we compute an observable quantity at a physical scale Q^2, the amplitude will have factors going as $\ln\Lambda^2/Q^2$. When we combine these with the coupling g_0^2, the cutoff dependence disappears, and the log becomes $\ln M^2/Q^2$, which diverges as $M \to 0$. Thus we can't necessarily take the continuum limit while keeping perturbation theory under control; it only works if Q^2 is larger than the characteristic scale of QCD, $\lambda^2_{\rm Politzer} \simeq (1\,{\rm GeV})^2$, where perturbation theory starts to break down.

Related to this, another possible source of confusion is the arbitrariness of the choice of M in (15.14). I showed that M is perfectly arbitrary since we can absorb any change of M into a redefinition of the c_i constants. That is true, but in practice, for a given application, some choices of M will be more advantageous than others. Generally, if we are going to use an observation at a particular scale Q^2 to fix the value of g_0^2, then it makes sense to choose $M^2 \sim Q^2$, to get rid of large logarithms $\ln M^2/Q^2$, that would also appear in the higher-loop diagrams that we are not including in our calculation. This choice reduces the error coming from these higher-order diagrams. This argument also shows why it would not make sense to try to take M^2 all the way to zero: perturbation theory in QCD is breaking down at scales $Q^2 \lesssim \lambda_P^2$. So we should not take $M < \lambda_P$. It is sometimes convenient to *define* M by imposing a choice like $c_0 = 0$ on the arbitrary constant, to make eq. (15.14) look nicer. Then M takes on a physical significance (within a particular regularization method), that can be called the scale where QCD is becoming nonperturbative. In this convention, $M = \lambda_P$, which has been determined to be around 200 MeV.

Because the coupling only runs logarithmically, it takes considerable experimental effort to reach energies where convergence of the perturbation expansion improves dramatically. And on the lattice, it takes enormous numerical effort. Consider that on a 4D lattice, by cutting the lattice spacing in half, we compound the computational problem by a factor of $2^4 = 16$, yet the coupling has decreased by a factor of only $1/\ln 4 \cong 0.7$. Thus it seems nearly hopeless to achieve quantitative accuracy on the lattice, given the limitations of computers, and it would be nice to come up with another nonperturbative method that is not just brute force.

16. THE RENORMALIZATION GROUP (1-7-88)

Last time we saw that the bare coupling could be expressed in a way that I will rewrite in the form

$$\begin{aligned}\frac{1}{g_0^2(\Lambda)} \;&=\; \beta_0 \ln\frac{\Lambda^2}{\lambda_P^2} + \frac{\beta_1}{\beta_0}\ln\left(\frac{\ln\Lambda^2}{\lambda_P^2}\right) + \text{const.} \\ &+\; \mathcal{O}\left(\frac{1}{\ln(\Lambda^2/\lambda_P^2)}\right)\end{aligned} \tag{16.1}$$

where λ_P is short for a scale I am calling "$\lambda_{\rm Politzer}$," and the last term, that vanishes as the cutoff goes to infinity, is scheme-dependent. If we choose g_0 in this way, for a given method of regularization, physics is unchanged from scheme to scheme and as the cutoff varies. If we rewrite the second term on the right-hand side of (16.1) as $(\beta_1/\beta_0)\ln\big(1/(\beta_0 g_0^2)\big)$, the terms that vanish as $\Lambda \to \infty$ are cleaner and have no $\ln\ln$ coefficients in the numerators of $1/\ln\Lambda^2$.[44]

Now we can differentiate,

$$\frac{d\left(\frac{1}{g_0^2(\Lambda)}\right)}{d\left(\ln\frac{\Lambda^2}{\lambda_P^2}\right)} = \beta_0 + \beta_1\,g_0^2 + \beta_2\,g_0^4 + \ldots \tag{16.2}$$

This is closely related to the beta function,

$$\beta(g) = -\frac{dg_0}{d\ln\Lambda} = \beta_0\,g_0^3 + \beta_1\,g_0^5 + \ldots \tag{16.3}$$

---

[44] This statement is justified in eq. (16.4).

Here β_0 and β_1 are scheme-independent, but β_2, $\beta_3, \ldots$ are scheme-dependent and they affect only the $\mathcal{O}(1/\ln\Lambda^2)$ terms. As I explained in the last lecture, we must choose the constant in (16.1) in order to define λ_P; so we made a specific choice. We define the constant to be zero in the $\overline{\rm MS}$ scheme of dimensional regularization, and we can do likewise in Pauli-Villars. This defines λ_P. It is an arbitrary choice which depends on the regularization method. Making this choice, we can write an exact formula

$$\frac{1}{g_0^2(\Lambda)} = \beta_0 \ln\frac{\Lambda^2}{\lambda_P^2} + \frac{\beta_1}{\beta_0}\ln\left(\frac{1}{\beta_0\, g_0^2}\right) \tag{16.4}$$

which is an implicit *definition* of $g_0^2(\Lambda)$. It satisfies the differential equation

$$\begin{aligned}\frac{d\left(\frac{1}{g_0^2(\Lambda)}\right)}{d\left(\ln\frac{\Lambda^2}{\lambda_P^2}\right)} &= \beta_0 + \beta_1\, g_0^2 + \frac{\beta_1^2}{\beta_0}\, g_0^4 + \ldots \qquad (16.5)\\ &= \frac{\beta_0}{1-\frac{\beta_1}{\beta_0}g_0^2} \qquad \begin{pmatrix}\beta_0 = 11 - \frac{2}{3}n_f\\ \beta_1 = 102 - \frac{38}{3}n_f\end{pmatrix}\end{aligned}$$

where n_f is the number of quark flavors. Notice that β_2 and higher coefficients are all determined by β_0 and β_1 in this definition of $g_0(\Lambda)$. For large Λ, the different choices one could make for these higher coefficients are not important.

As I mentioned in a previous lecture, in QED we have a problem when we try to do the analogous thing. Let $e^2_{\rm th}$ denote the physical, observed value of the coupling. Then

$$\frac{1}{e^2_{\rm th}} = \frac{1}{e_0^2(\Lambda)} + \frac{1}{3\pi}\ln\left(\frac{\Lambda^2}{m^2}\right) \tag{16.6}$$

where we have a plus sign instead of minus between the two terms. Hence

$$\frac{1}{e_0^2(\Lambda)} = \frac{1}{e^2_{\rm th}} - \frac{1}{3\pi}\ln\left(\frac{\Lambda^2}{m^2}\right) \tag{16.7}$$

As Λ gets large, $1/e_0^2(\Lambda)$ becomes negative, and we lose unitarity. This means we can't push QED to arbitrarily high energies, at least not greater than

$$\frac{\Lambda^2}{m^2} \sim e^{137}\,. \tag{16.8}$$

Grand unification is one solution to this problem.[45]

[45] In my notes I have parenthetically, "(Even just electroweak theory seems to fix it.)" perhaps referring to the modification of the QED beta function by the other standard model interactions.

16.1. Measuring g^2

Recall that we could measure g^2 through the ratio

$$R = \frac{\sigma_{e^+e^-\to\text{had.}}}{\sigma_{e^+e^-\to\mu^+\mu^-}}; \qquad (R-1) = \frac{g_0^2}{\pi} + a\, g_0^4 + \cdots \tag{16.9}$$

at high energy. We wanted something that didn't depend on the mass of the quarks, another example being qq scattering at high energies. In this way we could concentrate on determining g_0^2. We would also like our observable to be dimensionless, such as $Q^2\sigma$ since the cross section σ has dimensions of (length)$^{-2}$. To avoid extraneous dependences, we can imagine keeping the scattering angles fixed as we increase the energy scale.

In general, our calculation of this quantity will depend on g_0^2, Λ and Q; call it $D_{\rm theory}(g_0^2, \Lambda, Q)$. At first we consider g_0^2 to be independent of Λ—it's just a parameter. Now we want the physically measured value of the observable

$$D_{\rm phys.} = D(g_0^2(\Lambda), \Lambda, Q) \tag{16.10}$$

to be independent of Λ as $\Lambda \to \infty$ (and of course it must have a well-defined limiting value). Recall that the coupling is dimensionless, which you can see from the action,

$$\begin{aligned} S &= \int d^4x\, \frac{1}{g_0^2} F_{\mu\nu}F_{\mu\nu} \qquad (16.11)\\ &= \int d^4x\, \frac{1}{g_0^2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu - A_\mu \overset{\times}{,} A_\nu\right)^2\end{aligned}$$

and remembering that $[A] = 1/L$ hence $[F] = 1/L^2$. So $[F^2] = 1/L^4$ and since the action is dimensionless, $[g_0^2] = 1$: the coupling is dimensionless. Therefore $D_{\rm theory}(g_{00}^2, \Lambda, Q)$ can only depend on the dimensionless ratio Λ/Q if we keep g_{00}^2 fixed.[46] Hence we can write $D_{\rm theory}$ in the form $D_{\rm theory}(g_{00}^2, \ln\Lambda^2/Q^2)$. Similarly, our prediction for the physically observed value must take the form

$$D_{\rm phys.} = D\left(g_0^2(\Lambda), \ln\Lambda^2/Q^2\right) \tag{16.12}$$

and it *cannot depend on* Λ. This tells us how it depends on Q for large Q.

Suppose we ignore the β_1 term in $g_0^2(\Lambda)$, and the g_0^4 term in $(R-1)$. Then

$$\begin{aligned} D_{\rm theory}\left(g_{00}^2, \ln\frac{\Lambda^2}{Q^2}\right) &= \frac{g_{00}^2}{\pi} \qquad (16.13)\\ D_{\rm phys.}\left(g_0^2(\Lambda), \ln\frac{\Lambda^2}{Q^2}\right) &= \frac{g_0^2(\Lambda)}{\pi} = \frac{1}{\pi\beta_0\ln\frac{\Lambda^2}{\lambda_P^2}}\end{aligned}$$

[46] The subscript on g_{00}^2 is to distinguish the coupling g_{00}^2 that is considered to be independent of Λ from $g_0^2(\Lambda)$.

But this depends on Λ! The problem is that we didn't go to the next order:

$$\begin{aligned} D_{\rm theory}\left(g_{00}^2,\ln\frac{\Lambda^2}{Q^2}\right) &= \frac{g_{00}^2}{\pi}+g_{00}^4\left(\beta_1\ln\frac{\Lambda^2}{Q^2}+a_1\right)\\ D_{\rm phys.}\left(g_0^2(\Lambda),\ln\frac{\Lambda^2}{Q^2}\right) &= \frac{g_0^2(\Lambda)}{\pi} \qquad (16.14)\\ &= \frac{1}{\pi\beta_0\left(\ln\frac{\Lambda^2}{\lambda_P^2}-\ln\frac{\Lambda^2}{Q^2}\right)}\\ &= \frac{1}{\pi\beta_0\ln\frac{\Lambda^2}{\lambda_P^2}}+\frac{\ln\Lambda^2/Q^2}{\pi\beta_0\,\ln^2\Lambda^2/\lambda_p^2}\\ &+ \cdots \end{aligned}$$

where $\cdots$ represents terms of order g_0^4. The next order terms in $D_{\rm theory}$ must be

$$\frac{g_{00}^6}{\pi}\left(\beta_0^2\ln^2\frac{\Lambda^2}{Q^2}+2a_1\ln\frac{\Lambda^2}{Q^2}+a_2\right)$$

Terms of the form $(\beta_0\ln\Lambda^2/Q^2)^n$ are the leading logs. The fact that you can sum them is all due to demanding that $D_{\rm phys.}$ does not depend on Λ^2. If we do the sum then we get

$$\frac{1}{\pi\beta_0\ln\frac{Q^2}{\lambda_P^2}}+\frac{a_1}{\pi\left(\beta_0\ln\frac{Q^2}{\lambda_P^2}\right)^2}+\cdots \qquad (16.15)$$

where the first term comes from the leading logs, the second from the next-to-leading logs, and so on. Now we could write

$$(R-1)=\frac{\alpha(Q)}{\pi}+\frac{a_1}{\pi}\,\alpha^2(Q)+\cdots \qquad (16.16)$$

where

$$\alpha(Q)=\frac{1}{\beta_0\ln\frac{Q^2}{\lambda_P^2}} \qquad (16.17)$$

This shows that $\alpha(Q^2)$ operates like a coupling constant that depends on energy. We did this by neglecting β_1. If we keep β_1, we get

$$\frac{1}{\alpha(Q^2)}=\beta_0\ln\frac{Q^2}{\lambda_P^2}+\frac{\beta_1}{\beta_0}\,\ln\left(\frac{1}{\beta_0\,\alpha(Q^2)}\right) \qquad (16.18)$$

Evaluating it at different energies Q in GeV, using $\lambda_P = 0.2\,\text{GeV}$, gives[47]

Q	1	3	5	10	30	50	100	300	1000
n_f	3	3	4	4	5	5	6 ?	6 ?	6 ???
α	0.43	0.26	0.23	0.19	0.16	0.15	0.14	0.12	0.11 ± 0.01

[47] This table, which RPF apparently computed himself, was not given in the lecture, but it appears in his private notes for this lecture.

16.2. Renormalization group equations

Now we are ready to derive the renormalization group equations, using the fact that physical quantities cannot depend on Λ. $dD_{\rm phys.}/d\Lambda = 0$ implies that

$$\left.\frac{\partial D}{\partial g_{00}^2}\right|_{g_{00}^2=g^2(\Lambda)}\times\frac{dg^2(\Lambda)}{d\ln(\Lambda^2/\lambda_P^2)}-\frac{\partial D}{\partial\ln Q^2}=0 \qquad (16.19)$$

where

$$\begin{aligned}\frac{dg^2}{d\ln\Lambda^2} &= -g\,\beta(g) \qquad (16.20)\\ &= -\left[\beta_0 g^4(\Lambda)+\beta_1 g^6(\Lambda)+\beta_2 g^8(\Lambda)\cdots\right]\end{aligned}$$

Suppose we have worked out the theoretically predicted value

$$D_{\rm th}\left(g_{00}^2,\ln\frac{\Lambda^2}{Q^2}\right)=b_0+b_1g_{00}^2+b_2g_{00}^4+b_3g_{00}^6+\cdots \qquad (16.21)$$

where the b_m may depend on $t\equiv\ln\Lambda^2/Q^2$. Let $b_m' = db_m/dt$. Then the RG equation says that

$$\begin{aligned} b_0' &+ b_1'\,g_{00}^2+b_2'\,g_{00}^4+b_3'\,g_{00}^6\\ &= \left[\beta_0\,g_0^4(\Lambda)+\beta_1\,g_0^6(\Lambda)+\beta_2\,g_0^8(\Lambda)\cdots\right]\\ &\times\ \left[b_1+2b_2\,g_{00}^2+3b_3\,g_{00}^4\cdots\right] \qquad (16.22)\end{aligned}$$

This tells us that

$$b_0'=b_1'=0\,; \qquad (16.23)$$

hence b_0 and b_1 are constants, that we can calculate theoretically; call them c_0 and c_1. Moreover

$$\begin{aligned} b_2' &= c_1\,\beta_0 \quad\Longrightarrow\quad b_2=c_1\beta_0\ln\frac{\Lambda^2}{Q^2}+c_2\\ b_3' &= (b_1\beta_1+2b_2\beta_2)=c_1\beta_1+2c_1\beta_0\ln\frac{\Lambda^2}{Q^2}+2c_2\beta_0\\ &\Longrightarrow\ b_3=c_1\beta_0\ln^2\frac{\Lambda^2}{Q^2}+(2c_2\beta_0+c_1\beta_1)\ln\frac{\Lambda^2}{Q^2}+c_3 \end{aligned} \qquad (16.24)$$

etc. Then

$$\begin{aligned} D_{\rm phys.}\big|_{g_0^2(\Lambda)} &= c_0+c_1g_0^2(\Lambda)+\left(c_1\beta_0\ln\frac{\Lambda^2}{Q^2}+c_2\right)g_0^4(\Lambda)\\ &+\ \left(c_1\beta_0\ln^2\frac{\Lambda^2}{Q^2}+(2c_2\beta_0+c_1\beta_1)\ln\frac{\Lambda^2}{Q^2}+c_3\right)g_0^6(\Lambda)\\ &+\ \cdots \qquad (16.25)\end{aligned}$$

At each order n in perturbation theory, a new constant term c_n arises. But this new constant is typically less important than the preceding terms appearing in the coefficient of g_0^{2n}, that come with higher powers of logs. In other words, c_1 determines all the leading logs, c_1 and c_2 all the next-to-leading logs, *etc.* We can simplify (16.25)

because we know it doesn't depend on Λ. Organizing it in terms of the c_i coefficients we get

$$D_{\text{phys.}} = c_0 + c_1\,\alpha(Q^2) + c_2\,\alpha^2(Q^2) + c_3\,\alpha^3(Q^2) + \cdots \tag{16.26}$$

where $\alpha(Q^2)$ satisfies

$$\frac{d\alpha(Q^2)}{d\ln Q^2} = \beta_0\,\alpha^2 + \beta_1\,\alpha^3 + \beta_2\alpha^4 + \cdots \tag{16.27}$$

If one could measure D at such a high energy that $\alpha(Q^2)$ was small and the series converged rapidly, it would provide way to determine $\lambda_P \sim 200\,\text{MeV}$.

17. RENORMALIZATION: APPLICATIONS (1-12-88)

Renormalization is a confusing subject, and one factor contributing to the confusion is the proliferation of different conventions. I am guilty of this by my preferred normalization of the gauge coupling, which is not the same as that of the rest of the world. For your convenience, let me translate some previous results into the more conventional form, where the coupling and its associated fine-structure constant are related as

$$\alpha = \frac{g^2}{4\pi} \tag{17.1}$$

Then the running of the renormalized coupling, (16.18), takes the form

$$\frac{4\pi}{\alpha(Q^2)} = \beta_0\,\ln\frac{Q^2}{\lambda_P^2} + \frac{\beta_1}{\beta_0}\ln\frac{4\pi}{\beta_0\,\alpha(Q^2)} \tag{17.2}$$

Now let's review what we learned in the previous lecture, concerning the utility of this expression, that sums up the leading logarithmic dependences in the perturbation expansion. Namely, we can take an amplitude computed at tree level, and replace its α dependence by eq. (17.2), to resum the most important subclass of loop contributions to all orders, which improves the tree-level prediction. Furthermore, we can extend this to subleading contributions. Suppose we computed an amplitude at one-loop order and found a result going as

$$\mathcal{M} = \alpha_0 + \alpha_0^2\left(\ln\frac{\Lambda^2}{Q^2} + c\right) \tag{17.3}$$

The term with $\ln Q^2$ is already contributing to the leading logs that we obtain by replacing $\alpha_0 \to \alpha(Q^2)$ in the lowest order contribution. Thus the correct way to incorporate the next-to-leading logs is to replace

$$\mathcal{M} \to \alpha(Q^2) + c\,\alpha^2(Q^2) \tag{17.4}$$

I have avoided some of the complications of renormalization so far by only discussing gauge invariant, physical quantities. We could also consider the renormalization of more general quantities like Green's functions

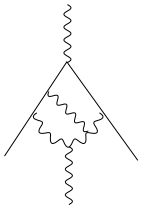
$$\langle A(x_1)\cdots A(x_n)\rangle\,. \tag{17.5}$$

Then it is not sufficient to talk about only the renormalization of the couplings, but also the wave function renormalization, that contributes to anomalous dimensions in the scaling of such a Green's function. I am not going to discuss these kinds of issues, but there are many references that do so, for example Renormalization by John Collins, Cambridge University Press, 1984. If you are only interested in physical, measurable quantities, these complications can be avoided.

One further point pertaining to the previous lecture is about the dependence of the beta function on the number of flavors. We have noticed that one of the nice features of QCD is its good behavior in the ultraviolet. This assumes there aren't too many flavors of quarks. There are five or six that we already know about, but there could be more—nobody knows why there should only be three families. But probably there are not 17 flavors of quarks, which is the critical number that would ruin the good UV behavior of QCD.

17.1. Power counting of divergences

Before embarking on explicit calculations of loop diagrams, it is enlightening to understand the general structure of divergences of the theory, and you are probably already familiar with this, but I would like to review it nevertheless. Consider some rather complicated diagram like



$$\sim \int \frac{d^4k\,d^4p}{(p^2,\ k^2,\ p\cdot k\ +\cdots)^7} \times f(p,\ k,\cdots) \tag{17.6}$$

We count 7 propagators, schematically indicated by the denominator of (17.5), 8 powers of momentum from the integration measure, and a numerator f that goes like (momentum)^5 from rationalizing the fermion propagators and counting the 3-gluon interactions. So according to power counting, this goes as (momentum)$^{-1}$ and is therefore superficially convergent. We call this exponent N_d, the superficial degree of divergence. It tells us that a given diagram generically behaves as

$$\begin{aligned} &N_d = 0,\ \text{log divergence}\\ &N_d = 2,\ \text{quadratic divergence}\\ &N_d < 0,\ \text{converges} \end{aligned} \tag{17.7}$$

We call it "superficial" because it is possible to construct exceptional cases in which there is a divergence even if

$N_d < 0$. This would be the case if the integrals somehow factorized into a product of one that was highly convergent times another that diverged. More typically however, we will see that this N_d often *over*estimates the degree of divergence, as a consequence of gauge symmetry.

Now it would be rather tedious to have to do this kind of counting for every possible diagram that may arise, but fortunately we don't have to. There is a beautiful topological relation that does it for us, solely in terms of the numbers of external lines of different kinds, independently of how complicated the diagram is, such as the number of loops. The relation is easiest to prove for a vacuum diagram with no external legs. It is a fact from topology that such a diagram satisfies

$$\#\text{ of Edges} + \#\text{ of Vertices} = \#\text{ of Faces} + 2 \qquad (17.8)$$

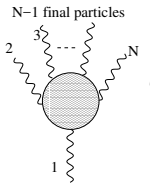
This is a relationship between the number of loops, the number of vertices and the number of propagators.

An easy way to arrive at the result is to first notice that a vacuum diagram, by dimensional analysis, must have $N_d = 4$. Imagine that we cut an internal quark line in a loop in such a diagram, to add two external quarks. We thereby remove one integration over loop momenta and one fermion propagator, which reduces N_d by 3. On the other hand, imagine adding an external gluon to some line on the diagram. It creates an extra fermion propagator, or else an extra gluon propagator with a coupling proportional to momentum in the numerator. Or we convert a triple-gluon vertex with dimension 1 into a dimensionless 4-gluon vertex. In any case, we reduce N_d by 1. Therefore it must be that

$$N_d = 4 - \mathcal{N}_g - \frac{3}{2}\mathcal{N}_q, \qquad (17.9)$$

where $\mathcal{N}_g$ ($\mathcal{N}_q$) is the number of external gluon (quark) lines.

Another way of deriving (17.9) is to use dimensional analysis for the general case. Let's illustrate this for some diagram with an arbitrary number of external particles, representing a process in which particle 1 decays into $N-1$ final state particles:



Rate of decays of 1 into $N-1$ particles:

$$d\Gamma = \frac{1}{2E_1}|T|^2 \prod_{i=2}^{N}(2\pi)\delta(p_i^2 - m_i^2)\frac{d^4p_i}{(2\pi)^4} \times (2\pi)^4\delta^{(4)}\left(p_1 - \sum_{i=2}^{N} p_i\right) \qquad (17.10)$$

I have drawn them as though they are all external gauge bosons, but we will also discuss the case when some of them are quarks. The shaded blob could contain any number of loops and internal lines; it doesn't matter how complicated the diagram is. Next to it I gave the formula for the differential decay rate, that we know has dimensions of mass or energy. Therefore we can determine the dimensions of the amplitude $|T|$ that corresponds to the diagram. By moving the factor $1/E_1$ to the other side of the equation, we get

$$\begin{aligned} [\text{energy}^2] &= [|T|^2]\,[\text{energy}]^{2(N-1)}\,[\text{energy}]^{-4} \\ \text{hence} \quad [|T|] &= [E]^{4-N} \end{aligned} \qquad (17.11)$$

where N is the total number of lines coming out, regardless of whether they are bosons or fermions.

If the amplitude T had the same dimensionality as the amputated loop diagram, we would conclude that $N_d = 4 - N$, since the coupling g is dimensionless, and so all dimensional factors are associated with momenta. In the case where all external particles were bosons, this would be the right answer. If some of them are fermions, it is not right because the external fermions have spinors associated with them, that have dimensions, and this makes the mass dimension of the diagram differ from N_d. But that is easy to correct for. You recall that summing the exterior product of two spinors over their polarizations gives the projection operator $\sum u\bar{u} = \not{p} + m$; therefore each spinor has dimension 1/2. This means we have to correct the previous result for N_d to read

$$N_d = 4 - \mathcal{N}_g - \mathcal{N}_q - \tfrac{1}{2}\mathcal{N}_q \qquad (17.12)$$

in agreement with (17.9).

I made a remark above, about gauge invariance, or possibly other symmetries, causing some diagrams to be more convergent than predicted by our formula for N_d. The most famous example is the case of $\mathcal{N}_g = 2$, which are diagrams contributing to the gluon vacuum polarization. Gauge invariance tells us that they should depend on the external gluon momentum q as

$$q^2\,\delta_{\mu\nu} - q_\mu q_\nu \qquad (17.13)$$

This means that two factors of momentum that we counted toward the degree of divergence are not loop momenta; instead they can be brought outside of the integral, making it more convergent that we naively estimated. To remind ourselves of this possibility, we could add an extra term $-\mathcal{P}$, so that

$$N_d = 4 - \mathcal{N}_g - \tfrac{3}{2}\mathcal{N}_q - \mathcal{P} \qquad (17.14)$$

where $\mathcal{P}$ is the known power of the coefficient in front of the integral.

Exercise. Show that the power of couplings g^P of an arbitrary diagram is given by

$$P = 2 \times (\#\text{ of loops}) + \mathcal{N}_g + \mathcal{N}_q - 2 \qquad (17.15)$$

This is probably easiest to do in the usual normalization of the fields, where the couplings appear in the conventional way, g for the 3-particle vertices and g^2 for the 4-gluon vertex.

A fortunate consequence of the formula (17.14) is that only a finite number of the different kinds of diagrams are divergent, when we classify them by their numbers of

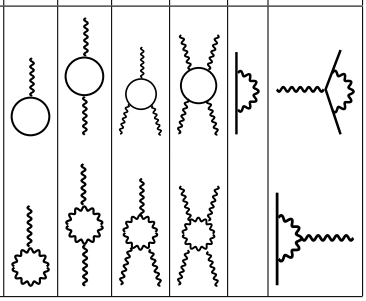
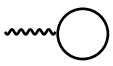
$\mathcal{N}_g$	1	2	3	4	0	1
$\mathcal{N}_q$	0	0	0	0	2	2
N_d	3	2	1	0	1	0
$\mathcal{P}$		2	1	0	1	0
typical diagrams						

Table 1: Superficial degrees of divergence, N_d (before accounting for $\mathcal{P}$), for graphs with $\mathcal{N}_q$ external quarks and $\mathcal{N}_g$ external gluons. $\mathcal{P}$ is the power of external momentum factors.

external lines. Therefore we can make a table to illustrate all the possibilities:

The tadpole diagrams, with $\mathcal{N}_g = 1$, provide another example of our statement that symmetries can make a diagram more convergent than power counting would suggest,


$$\sim \langle A_\mu \rangle \,, \tag{17.16}$$

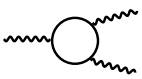
This diagram is naively divergent with $N_d = 1$, but in fact it vanishes. One can think of it as an expectation value of the gluon field. Such a thing, if nonzero, would spoil Lorentz invariance, as well as gauge invariance. And it would break discrete symmetries like C and P.

We already discussed the vacuum polarization diagram, the fact that it is proportional to


$$\sim \delta_{\mu\nu} q^2 - q_\mu q_\nu \tag{17.17}$$

We will derive this result later on. It is an example of how one must be careful about regulating the divergences from the loops in a gauge invariant way. The behavior (17.17) reduces its N_d by 2, so that instead of being quadratically divergent, it is only logarithmically divergent, but if the regularization method failed to respect gauge invariance, it would be afflicted with this more severe quadratic divergence.

Moreover, symmetry prevents the diagram


(17.18)

from being linearly divergent as its $N_d = 1$ would suggest. The only kind of Lorentz-invariant loop integrand one could write, consistent with this power counting, has the form $p \cdot X/(p^2)^2$, schematically, where p is the loop momentum and X represents an external momentum or polarization vector. But $p_\mu/(p^2)^2$ is odd under $p_\mu \to -p_\mu$, so its integral must vanish, as long as the regularization procedure does not introduce any pathology that would spoil this reasonable expectation. And we can also argue that it must vanish by Lorentz symmetry, since $\int d^4p\, p_\mu/(p^2)^2$ would define some preferred direction in spacetime if it were nonzero. Therefore the 3-gluon amplitudes are also only logarithmically divergent. In fact gauge invariance provides yet another reason this must be so: we know that the three-gluon interaction comes with a power of external momentum, from its Feynman rule, and this explains why the actual form of the integrand must be $q_\mu/(p^2)^2$, giving $N_d = 0$.

There is one possible caveat to the gauge invariance argument that should be kept in mind however. There is no guarantee that individual diagrams will be gauge invariant; only the sum of all diagrams contributing to a given process at a given order must necessarily be gauge invariant.

17.2. Choice of gauge

Before we embark on explicit calculations of loop diagrams, I wanted to discuss the relative advantages of some choices of gauge relative to others, when it comes to defining the gluon propagator. In our earlier discussion of gauge fixing, lecture 12, we discussed a particular class of gauges $\eta \cdot A = 0$, involving an arbitrary four vector η_μ. This is actually not the most convenient one for doing perturbative calculations, even though it was conceptually appealing. For one thing, it spoils Lorentz invariance temporarily, although these terms must cancel out in the end.

A simpler choice would be the Lorentz gauge $\partial_\mu A_\mu = 0$. Let us recall how the Faddeev-Popov procedure would work in this case. The gauge-fixed path integral takes the form

$$Z = \int e^{i\int \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu - A_\mu \overset{\times}{} A_\nu)^2}\, \Delta(A)\, \delta[\partial_\mu A_\mu]\, \mathcal{D}A_\mu \,, \tag{17.19}$$

omitting for simplicity the quarks. We rewrite the determinant $\Delta(A)$ as a path integral over ghost fields. We take advantage of the fact that this determinant does not change at all if we impose a slightly different choice of gauge, $\partial_\mu A_\mu = f(x)$, with an arbitrary function f. Therefore we are free to do a weighted average over the path integral,

$$Z \to \int \mathcal{D}f\, e^{\frac{i}{2}\int f^2 d^4x}\, Z \tag{17.20}$$

Then the delta functional gets rid of $\int \mathcal{D}f$, and we are left with a new term $\frac{1}{2}(\partial_\mu A_\mu)^2$ in the Lagrangian, that allows the propagator to be defined. To see this, consider the modified equation of motion for the gauge field, including a source term,

$$\Box A_\nu - \cancel{\partial_\nu \partial_\mu A_\mu} = S_\nu \tag{17.21}$$

and notice that the crossed-out term is removed by the new gauge-fixing term, and therefore we may invert the $\Box$ operator and solve for the gauge field,

$$A_\nu = \frac{1}{\Box} S_\nu = \frac{1}{k^2} S_\nu \,, \tag{17.22}$$

which shows that the propagator is simply $\delta_{\mu\nu}/k^2$ in this gauge. That obviously simplifies many perturbative computations, compared to the axial gauge propagator.

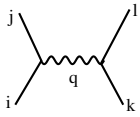
Of course nothing obliges us to choose $e^{\frac{i}{2}\int f^2 d^4x}$ as the weighting factor. One can equally well take $e^{\xi\frac{i}{2}\int f^2 d^4x}$ with some arbitrary number $\xi \neq 0$. This yields a more general class of covariant propagators of the form

$$P_{\mu\nu} = \frac{1}{k^2}\left(\delta_{\mu\nu} - \eta\frac{k_\mu k_\nu}{k^2}\right) \tag{17.23}$$

where η is related in some simple way to ξ.
Exercise. Find the relation between η and ξ.
The choice $\eta = 0$ is known as Feynman gauge. Another very convenient choice is $\eta = 1$, the Landau gauge. It has the property of being transverse, $k_\mu P_{\mu\nu}(k) = 0$, which leads to some simplifications in loop calculations. For example it greatly reduces the number of diagrams in the process we are going to consider next.[48] Nowadays you can find computer programs that will do the symbolic algebra for you, for computing such diagrams.[49] Nevertheless, it is much easier to avoid mistakes if you can reduce the number of diagrams.

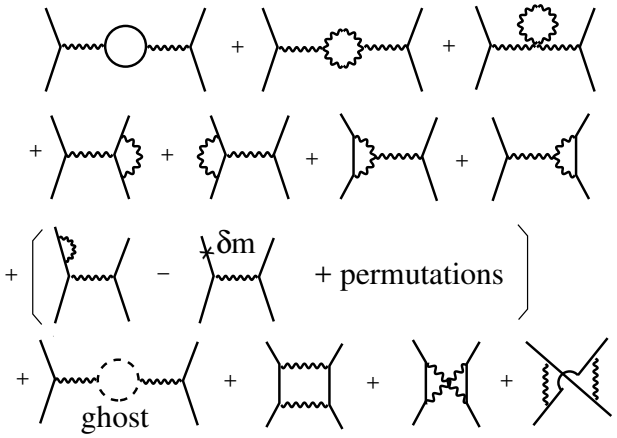
17.3. Explicit loop calculations

So far we have made numerous statements and expositions of a rather general nature, without getting into the details of computing loop diagrams. I would now like to go over some of those details, just to illustrate the calculational techniques. The example I will consider is the scattering of two quarks. To lowest order, as we know, it looks like

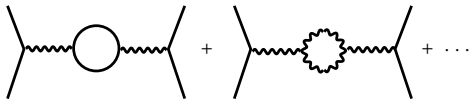


$$\sim \frac{g_0^2}{q^2}\,\frac{\lambda^A_{ji}}{2}\,\frac{\lambda^A_{lk}}{2} \tag{17.24}$$

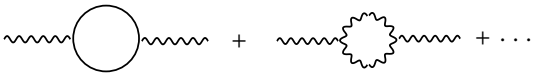
When we go to the next order, there are quite a few diagrams, including



One simplification we can immediately make is to first isolate the primitive divergences. For example, consider the diagrams of the form

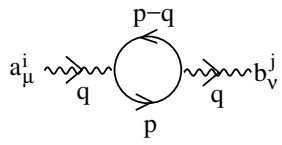


The interesting part of this calculation is the loop, not the external currents nor the gluon propagators that connect them to the loop. We might as well calculate the diagrams



by themselves, since it is trivial to take that result and add to it all the tree-level parts such as the gluon propagators and the spinors for the external currents.

Now to illustrate the techniques, I am just going to compute the simplest and dullest of all of these, namely the quark loop contribution to the vacuum polarization,



Once you understand the principles, it is just a matter of tedious effort to compute the harder ones.[50] By applying the rules, we can write down the expression for this diagram,

$$g_0^2\left(-\mathrm{Tr}\int \frac{d^4p}{(2\pi)^4}\, b^j_\nu \frac{1}{\not p - \not q - m}\frac{\lambda_j}{2}\gamma_\nu \frac{1}{\not p - m}\frac{\lambda_i}{2}\gamma_\mu a^i_\mu\right) \tag{17.25}$$

[48] In the lecture RPF says it reduces the number from 17 to 4. Perhaps he had in mind that the number of terms in the 3-gluon vertex is greatly reduced when taking only the transverse terms. Also the ghosts decouple in Landau gauge.

[49] Wolfram's SMP (Symbolic Manipulation Program), the forerunner of Mathematica, was in use at Caltech at this time.

[50] RPF mentions in the transcript that it is not so straightforward to get a gauge invariant result for diagrams with gluon loops, using Pauli-Villars regularization.

where I am taking a^i_μ and b^j_ν to be combined spin-color polarization vectors for the external gluons, but if you prefer you could replace these by spin polarization vectors ϵ_μ and ϵ_ν, and consider the gluon colors to be simply i and j. The trace here is a sum over all possible intermediate states, both spins and colors. Therefore it is really the product of two traces, one for the Dirac matrices and one for the color matrices.

Next, we should rationalize the quark propagators—multiply numerator and denominator by $(\not{p}+m)$—and carry out the traces. This looks like

$$\underbrace{\mathrm{tr}\left[\frac{1}{\not{p}-\not{q}-m}\gamma_\nu\frac{1}{\not{p}-m}\gamma_\mu\right]}\quad\underbrace{\mathrm{tr}\left(\frac{\lambda_i\lambda_j}{4}\right)} \tag{17.26}$$

$$\mathrm{tr}\left[\frac{(\not{p}-\not{q}+m)\gamma_\nu(\not{p}+m)\gamma_\mu}{[(p-q)^2-m^2][p^2-m^2]}\right]\qquad\frac{\delta_{ij}}{2}$$

$$=2\frac{\Big[(p-q)_\mu p_\nu+(p-q)_\nu p_\mu-\delta_{\mu\nu}(p\cdot(p-q)-m^2)\Big]}{[(p-q)^2-m^2][p^2-m^2]}\delta_{ij}$$

Notice that the gluon color is conserved by the loop.

Now we are left with the integral, that has exactly the same form as in QED. There is a famous trick for combining the denominators, that I adapted from Schwinger by eliminating a step from his Gaussian integral method,[51]

$$\int_0^1\frac{dx}{[ax+b(1-x)]^2}=\frac{1}{ab} \tag{17.27}$$

This allows us to combine the propagators into the form

$$\frac{1}{(p^2-2p\cdot q+q^2-m^2)(p^2-m^2)}= \int_0^1 dx\,\frac{1}{[p^2-2p\cdot qx+q^2x-m^2]^2} \tag{17.28}$$

so that the loop integral becomes

$$\int_0^1 dx\int\frac{d^4p}{(p^2-2p\cdot qx+q^2x-m^2)^2} \tag{17.29}$$
$$\times\quad\big(2p_\mu p_\nu-p_\mu q_\nu-p_\nu q_\mu-\delta_{\mu\nu}(p^2-p\cdot q-m^2)\big)$$

To further simplify it, we wish to complete the square in the denominator, by shifting the integration variable by $p\to p+qx$:

$$\int_0^1 dx\quad\int\frac{d^4p}{(p^2+q^2x(1-x)-m^2)^2} \tag{17.30}$$
$$\times\qquad\Big(2p_\mu p_\nu+(p\cdot q\text{ terms})+2q_\mu q_\nu\,x(1-x)$$
$$-\ \delta_{\mu\nu}(p^2-q^2x(1-x)-m^2+p\cdot q\text{ terms })\Big)$$

Once it is in this form, it is not necessary to keep careful track of the $p\cdot q$ terms in the numerator, since they are odd in p and integrate to zero.

[51] In my notes at this point I have written "stolen from Schwinger by eliminating a step from his Gaussian integral method."

17.4. Regularization

Everything so far seems perfectly innocuous and standard, but if we want to be careful, you will notice that I have cheated. The integral is divergent, so how do we know that the step of shifting the integration variable by $p\to p+qx$ is legitimate? It could conceivably change the result in some unphysical way, unless we have carefully defined what we mean by this integral. To be rigorous, we must specify exactly how we are going to regularize the integral, to cut off the ultraviolet divergence.

The method I want to use in this lecture is the historical one, invented by Pauli and Villars. As I alluded earlier, it turns out to be too simplistic to change the propagator as in (15.10). This was attempted by some of the early workers in the field, and it was found to spoil gauge invariance. Instead, one needs to apply this prescription to the *whole amplitude*. The consistent way is to replace the quark mass in the denominators by $m^2\to m^2+\Lambda^2$, and subtract the resulting expression from the original amplitude. The new integrand obtained in this way has good behavior in the ultraviolet, and so the procedure of shifting the integration variable is perfectly consistent, and moreover it preserves the gauge invariance, as we will see. And it is also consistent with our assumption that integrals like

$$\int d^4p\,\frac{p_\mu}{p^2-m^2}=0 \tag{17.31}$$

should vanish, with the understanding that this is now a shorthand for the fully regulated expression, where we have subtracted the corresponding term with $m^2\to m^2+\Lambda^2$.

The statement (17.31) looks trivial, but we can use it to derive a more interesting result, now that we are confident that shifts in the integration variable are legitimate. By shifting $p\to p-a$, we obtain

$$\int d^4p\,\frac{p_\mu}{(p-a)^2-m^2}\ =\ a_\mu\int\frac{d^4p}{(p-a)^2-m^2}$$

And then by differentiating with respect to a_ν and setting $a_\mu=0$, we get the useful identity

$$\int d^4p\,\frac{\delta_{\mu\nu}(p^2-m^2)-2p_\mu p_\nu}{(p^2-m^2)^2}=0 \tag{17.32}$$

Incidentally, this could also be obtained more directly, using

$$\int d^4p\,\frac{\partial}{\partial p_\nu}\frac{p_\mu}{p^2-m^2}=0$$

which should be true for the integral of the derivative of anything. We could be suspicious of such a statement in the unregulated theory, since the surface term might fail to vanish, but it rigorously vanishes in the regulated theory.

Let us now return to the calculation we started above, the computation of the vacuum polarization diagram. We want to reorganize the numerator of (17.30) so that it has one term in the same form as (17.32):

$$\int \frac{d^4p}{(p^2+q^2x(1-x)-m^2)^2}\Big[2p_\mu p_\nu - \delta_{\mu\nu}(p^2+q^2x(1-x)-m^2) - 2q_\mu q_\nu x(1-x) + 2\delta_{\mu\nu}q^2x(1-x)\Big]$$

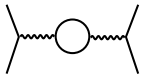
This has the pleasing feature that the first term, which appears to be quadratically divergent, actually vanishes. We are left with the second term, having the form (17.13) that I said must arise as a consequence of the gauge symmetry:

$$2(q_\mu q_\nu - \delta_{\mu\nu}q^2)\int_0^1 x(1-x)dx \int \frac{d^4p}{(p^2+q^2x(1-x)-m^2)^2} \tag{17.33}$$

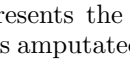
This leaves the logarithmically divergent integral, whose evaluation I will take up in the next lecture.

18. RENORMALIZATION, CONTINUED (1-14-88)

To remind you, we were computing the one-loop corrections to quark-quark scattering, and had noted that it is convenient to factorize the amplitude with the gluon vacuum polarization correction in the form



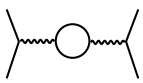
$$= J^i_\mu\, g_0^4\, \frac{1}{q^2}\, B\, \frac{1}{q^2}\, J^i_\mu \tag{18.1}$$

where B represents the simpler diagram  with the gluon lines amputated, and J^i_μ are the external quark currents, with color indices i. We found that B takes the form

$$B = 2(\delta_{\mu\nu}q^2 - q_\mu q_\nu)I, \tag{18.2}$$

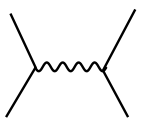
$$I = \int_0^1 dx\, x(1-x)\int \frac{d^4p/(2\pi)^4}{[p^2-m^2+q^2x(1-x)]^2}$$

Because the currents are conserved, $q\cdot J = 0$, we can simplify (18.1) slightly,



$$= 2J^i_\mu\, g_0^4\, \frac{I}{q^2}\, J^i_\mu \tag{18.3}$$

which has the same form as the tree-level contribution,



$$= J_\mu \frac{g_0^2}{q^2} J_\mu \tag{18.4}$$

Written in this way, it is clear that the loop contribution can be expressed as a change in the coupling constant,

$$g_0^2 \to g_0^2 + 2g_0^4 I \to \frac{g_0^2}{1-2g_0^2 I} \equiv g^2_{\rm eff} \tag{18.5}$$

where the first arrow indicates a result that is consistent to the order of perturbation theory at which we are working, while the second one uses the hindsight of resumming the leading logs, that we discussed in the last lecture.

Now to do the logarithmically divergent integral I, I will continue to use Pauli-Villars regularization, although later on we will introduce the more elegant method of dimensional regularization. Hence we must subtract from I the similar quantity with the modified propagator

$$\frac{1}{(p^2-(m^2+\Lambda^2)+q^2x(1-x))^2} \tag{18.6}$$

Rather than directly subtracting, there is a nicer way to implement this, by first thinking of I as a being a function of m^2,

$$I(m^2) = \int_0^1 dx\, x(1-x)\int \frac{d^4p/(2\pi)^4}{[p^2-m^2+q^2x(1-x)]^2} \tag{18.7}$$

and then differentiating with respect to m^2. Doing this makes the integral convergent, even without subtracting anything. If we then integrate I' with respect to m^2,

$$-\int_{m^2}^{m^2+\Lambda^2} I'(M^2)\, dM^2 = I(m^2) - I(m^2+\Lambda^2) \tag{18.8}$$

the result is the subtraction we originally wanted to carry out. The trick then is to evaluate the convergent integral appearing in $I'(M^2)$, and postpone doing the integral over M^2 until afterwards:

$$\int_{m^2}^{m^2+\Lambda^2} dM^2 \int_0^1 dx\, x(1-x) \underbrace{\int \frac{d^4p/(2\pi)^4}{[p^2+q^2x(1-x)-M^2]^3}}$$

$$\equiv \int \frac{d^4p}{(2\pi)^4}\frac{1}{(p^2-L)^3} = \frac{1}{32\pi^2 i L} \tag{18.9}$$

In performing the integral, I have glossed over a few steps that I assume you are already familiar with, from a previous course on quantum field theory, notably doing the Wick rotation to avoid the poles from the $i\epsilon$ prescription, which gives the factor of $1/i$.

Next we carry out the integral over M^2,

$$\int_{m^2}^{m^2+\Lambda^2} dM^2 \;\; \frac{1}{16\pi^2 i}\,\frac{1}{M^2-q^2x(1-x)} = \frac{1}{16\pi^2 i}\ln\left(\frac{m^2+\Lambda^2-q^2x(1-x)}{m^2-q^2x(1-x)}\right). \tag{18.10}$$

Since we are ultimately interested in the limit as $\Lambda \to \infty$, this can be simplified by ignoring the finite terms in the numerator of the argument of the logarithm. As we discussed before, the $\ln\Lambda$ divergence gets absorbed into the tree-level contribution by redefining the bare coupling g_0.

To further simplify the discussion, I would like to consider momenta such that $-q^2 \gg m^2$, so that we can ignore the quark mass. You might be concerned that this could give rise to an infrared divergence from the places where $x = 0$ or 1 when we perform the integral over x, but because the integrand is a log, these singularities are integrable and lead to no difficulty. In this approximation, we have

$$\int \frac{dx\, x(1-x)}{16\pi^2 i}\left[\ln\left(\frac{\Lambda^2}{-q^2}\right) - \ln x - \ln(1-x)\right] = \left(\frac{1}{6}\ln\left(\frac{\Lambda^2}{-q^2}\right) - \frac{5}{18}\right) \qquad (18.11)$$

Remember that $-q^2 \equiv Q^2$ is positive, since $\vec{q}$ is the momentum transfer in the electron-electron scattering. The integral of the logarithm can be done using integration by parts.

Now we have evaluated the integral I, and we can put it back into the expression (18.3):

$$J\frac{1}{Q^2}J\left[g_0^2 + \frac{g_0^4}{16\pi^2}\left(\Delta\beta_0\left(\ln\frac{\Lambda^2}{Q^2}\right) + a\right)\right] \qquad (18.12)$$

$$\text{where } \Delta\beta_0 = -\frac{2}{3}n_f \quad \text{and } a = \frac{10}{9}n_f \qquad (18.13)$$

where n_f is the number of quark flavors having mass less than Q^2; otherwise our approximation $Q^2 \gg m^2$ is not valid. Here $\Delta\beta_0$ is just the quark contribution to β_0; the full β_0 gets an additional contribution of +11 from the gluon loop, that we are not calculating here.

We can now see more explicitly how the renormalization of the bare coupling is derived, which absorbs the dependence on the cutoff arising from the loop diagram. Consider the effective coupling defined in (18.5),

$$g_{\rm eff}^2 = \frac{g_0^2}{1 - \frac{g_0^2}{16\pi^2}\beta_0 \ln\frac{\Lambda}{Q^2}} = \frac{16\pi^2}{\left(\frac{16\pi^2}{g_0^2}\right) - \beta_0 \ln\frac{\Lambda^2}{Q^2}}\,. \qquad (18.14)$$

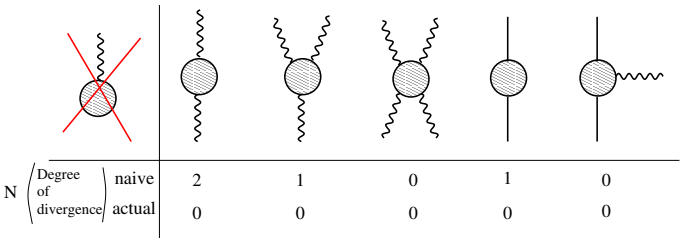
The second line makes it clear how g_0^2 must depend on Λ in order that $g_{\rm eff}$ be independent of Λ,

$$\frac{16\pi^2}{g_0^2(\Lambda)} \equiv \frac{1}{\hat{g}_0^2(\Lambda)} = \beta_0 \ln\frac{\Lambda^2}{\lambda_P^2} + c_1 \qquad (18.15)$$

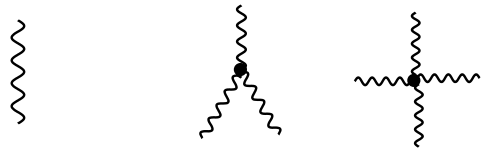
where I have inserted a renormalization scale λ_P, since g_0 is a Lagrangian parameter that cannot depend on the external momentum Q. As I previously mentioned, we are free to choose a convention for defining λ_P such that the arbitrary constant c_1 vanishes, if so desired.

18.1. Effective Lagrangian perspective

Although we imagined that the value of g_0^2 has been fixed in this example by comparing to a particular observable, the scattering cross section for two quarks, it is important to emphasize that once $g_0^2(\Lambda)$ has been determined, it is now valid for the study of any process; we do not need to define a separate $g_0^2(\Lambda)$ for every different observable. One way to understand this is from the fact that there is a finite number of primitively divergent diagrams in the theory, repeated in this table,



You recall that the tadpole diagram vanishes. The renormalizability of the theory implies that all the divergent diagrams can be related to an effective Lagrangian contribution that has the same form as the bare Lagrangian. So for example, all the divergent diagrams with only external gluons must correspond to terms in the Lagrangian in this manner:



$(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \qquad A_\nu^\times A_\mu^\bullet \partial_\mu A_\nu \qquad A_\nu^\times A_\nu^\bullet A_\mu^\times A_\nu$

A priori, we would have to renormalize three different parameters to absorb the divergences. But because of gauge invariance, we know that they must organize themselves into the form $\mathrm{tr}\, F_{\mu\nu}F_{\mu\nu}$, that depends only on the single parameter g_0. Therefore these divergences are related to each other in such a way that they can all be absorbed by the renormalization of the single parameter g_0. This is perhaps easiest to see in the convention where we keep g_0 out of the field strength definition and put it as a prefactor in the Lagrangian, $(1/g_0^2)\mathrm{tr}\, F_{\mu\nu}F_{\mu\nu}$. Then the three divergences indicated above would all contribute to the shift in the effective Lagrangian that goes as

$$\ln\Lambda^2\, \mathrm{tr}\, F_{\mu\nu}F_{\mu\nu} \qquad (18.16)$$

This of course assumes that the regularization method did not spoil gauge invariance. Otherwise we would have to fudge the results to make this work out.

A similar argument applies to the terms with external quarks. The self-energy diagram requires us to renormalize an additional quantity, the quark mass. But the vertex correction does not need anything new; gauge symmetry guarantees that the same renormalization of g_0 as needed for the gluons also suffices for the coupling to quarks.

To make this clearer, let's rephrase these statements in the context of the path integral. We could imagine, if we were sufficiently adept, being able to carry out the

integral over quarks for a fixed gauge field background,

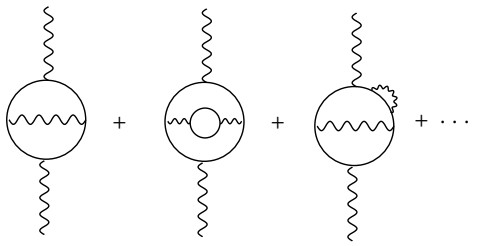
$$\int \mathcal{D}A_\mu\, e^{\,i\frac{1}{g_0^2}\int F_{\mu\nu}F_{\mu\nu}} \int \mathcal{D}\psi\mathcal{D}\bar\psi\, e^{\,i\int \bar\psi(i\not{D}-m)\psi}$$
$$= \int \mathcal{D}A_\mu\, e^{\,i\frac{1}{g_0^2}\int F_{\mu\nu}F_{\mu\nu}} \underbrace{\det\left(i\not{D}-m\right)}_{G(A)\,\sim\, e^{\,i\ln\Lambda^2\int F_{\mu\nu}F_{\mu\nu}}} \tag{18.17}$$

giving a functional determinant of the covariant Dirac operator, that I am calling $G(A)$. Even though we can't calculate $G(A)$ exactly, from perturbation theory we know that it has a divergent contribution like I have indicated, which has the same form as the tree-level gluon action. We can then rewrite the bare coupling in terms of a renormalized coupling, which is finite as $\Lambda\to\infty$, plus a correction designed to cancel the $\ln\Lambda$ divergences. In the convention I have chosen above, it would be easier to think of it as a correction to $1/g_0^2$:

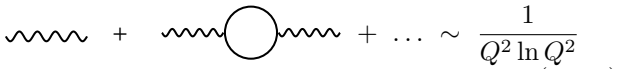
$$\frac{1}{g_0^2} = \frac{1}{g^2_{\rm ren}} + \delta\left(\frac{1}{g_0^2}\right) \tag{18.18}$$

These extra terms labeled as $\delta(1/g_0^2)$ are known as the *counterterms*.

There is an interesting consequence of the running for the convergence properties of the loops when we start to think about the higher-order contributions, such as



One can think of such diagrams as though they were at one order lower in perturbation theory, but constructed from propagators that have already been dressed at one loop,



$$\sim \frac{1}{Q^2\ln Q^2} \tag{18.19}$$

An integral that is normally considered to be logarithmically divergent would instead behave like

$$\int^\Lambda \frac{dp\,p}{p^2\ln p^2} \sim \ln(\ln\Lambda^2) \tag{18.20}$$

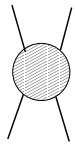
This is still divergent, but more mildly so. And going to higher order, one could get higher powers of logs in the denominator, which would make the integral converent,

$$\int_c^\Lambda \frac{dp\,p}{p^2\ln^2 p^2} \sim \int_c^\Lambda \frac{d\ln p^2}{\ln^2 p^2} \sim \frac{1}{\ln c} - \frac{1}{\ln\Lambda} \tag{18.21}$$

This could be understood as a consequence of replacing the vertices at the ends of the dressed propagator by the dressed vertices, that behave as $\alpha(Q)\sim 1/\ln(Q^2)$ to give this more convergent behavior.

18.2. Misconceptions

I would like to discuss a different viewpoint of the running coupling, that you may encounter in the literature, and that I consider to be misguided. The idea is to choose some physical amplitude—suppose for simplicity that at tree level it is linear in α—and to consider it as a function of Q. One could then *define* a running coupling $\underline{\alpha}(Q^2)$ to be exactly determined by this physical observable. An example would be the amplitude for the scattering of two quarks, to all orders in perturbation theory,



$$\equiv \frac{\underline{\alpha}(Q^2)}{Q^2} \tag{18.22}$$

However there would be no such thing as perturbation theory in regard to this particular process, since there is nothing to expand in: $\underline{\alpha}(Q^2)$ is the exact result. Another example is the correction $R-1$ for the process $e^+e^-\to$ hadrons that we discussed previously. It gives rise to a different definition of the coupling, call it

$$\underline{\underline{\alpha}}(Q^2) = \pi(R-1) \tag{18.23}$$

If we were to compare these two definitions, we would find out that they approximately agree, and a useful way to compare them would be by differentiating and trying to reconstruct the beta function. We would find that both definitions satisfy equations of the form

$$\frac{d\underline{\alpha}}{d\ln Q^2} = \beta_0\underline{\alpha}^2 + \beta_1\underline{\alpha}^4 + \beta_2\underline{\alpha}^6 + \dots \tag{18.24}$$

and that the first two coefficients β_0, β_1 agree for both definitions. But beyond that, the remaining coefficients are in general different between the two definitions, and unrelated to the values of β_0, β_1. Contrast that to the definition I made,

$$\frac{d\alpha}{d\ln Q^2} = \frac{\beta_0\alpha^2}{1-\frac{\beta_1}{\beta_0}\alpha^2} = \beta_0\alpha^2 + \beta_1\alpha^4 + \frac{\beta_1}{\beta_0}\alpha^6 + \dots \tag{18.25}$$

where the higher coefficients are all determined. In our procedure, the amplitudes for the two processes have to both be calculated perturbatively in α. This is how physics should work: we have a definite theory that is independent of the process and we predict the observable from it. The theory should not be predicated on one particular process or another.

Adding to the confusion caused by such proposals is the misconception that there is a momentum-dependent coupling constant in the Lagrangian. As I explained before, the Q dependence in $\alpha(Q)$ is just a shorthand to remind us how the loop-corrected amplitude depends on Q, that we can deduce by replacing $\lambda_P\to Q$ in the definition of $g_0(\Lambda)$. But the actual coupling that goes into the Lagrangian is $g_0(\Lambda)$, which does not depend on Q.

18.3. Dimensional regularization

Let me finish by giving a preview of how the logarithms arise in dimensional regularization, that we will discuss in more detail in the next lecture. Because the action must be a dimensionless quantity, when we continue the dimension of spacetime to some value $D = 4 - \epsilon$, the coupling that was dimensionless in $D = 4$ is no longer so. For this discussion I will adopt the field normalization where $F \sim \partial A + [A, A]$ and the coupling constant appears in front of F^2. Because of this form of F, there is no choice but to say that the dimensions of A are $[A] = 1/[L] = [M]$, and therefore the action looks like

$$\frac{1}{g^2}\underbrace{\int F_{\mu\nu}^2\, d^{\,D}x}_{\text{dimensions of M}^{4-D} = [M^\epsilon]} \tag{18.26}$$

Therefore to make the action dimensionless, we must have

$$[g^2] = [M^\epsilon] \tag{18.27}$$

To make this explicit, it is convenient to relate g to a dimensionless coupling g_0 and a mass scale that I suggestively call λ_P,

$$g^2 = g_0^2 \lambda_P^\epsilon \tag{18.28}$$

Now imagine redoing the calculation that led to eq. (18.5) using dimensional regularization. The result takes the form

$$\begin{aligned} g^2 + g^4 Q^{-\epsilon}\left(\frac{2\beta_0}{\epsilon}\right) &\to \frac{1}{\frac{1}{g^2} - \frac{2\beta_0 Q^{-\epsilon}}{\epsilon}} \\ &\approx \frac{1}{\frac{2\beta_0}{\epsilon}\lambda_P^{-\epsilon} - \frac{2\beta_0}{\epsilon}Q^{-\epsilon}} \end{aligned} \tag{18.29}$$

in which the $\ln \Lambda^2$ divergences of Pauli-Villars get replaced by $1/\epsilon$ poles. Here I have resummed the leading logs and observed that necessarily $g_0^{-2} = 2\beta_0/\epsilon + \text{const.}$, in order to cancel the divergence. Then we notice that

$$\frac{2}{\epsilon}(\lambda_P^{-\epsilon} - Q^{-\epsilon}) = \ln(Q^2/\lambda_P^2) + O(\epsilon) \tag{18.30}$$

just like the outcome of the cutoff method of regularization.

19. RENORMALIZATION (CONCLUSION); LATTICE QCD (1-19-88)

I would like to add something to our previous discussion concerning the renormalization group equation. Recall that the definition

$$\frac{16\pi^2}{\beta_0 g^2(\Lambda^2)} - \frac{\beta_1}{\beta_0^2}\ln\frac{16\pi^2}{\beta_0 g^2(\Lambda^2)} = \ln\frac{\Lambda^2}{\lambda_P^2} \tag{19.1}$$

tells us (in an implicit way) how the coupling g must depend on the cutoff Λ. Now imagine some physical process, whose amplitude $\mathcal{M}$—for definiteness I will pick an example where it is dimensionless—we compute from the theory with the cutoff, using the coupling g_0. $\mathcal{M}$ is generically a function of g_0, some momenta which for simplicity I will represent by a single scale Q, and the cutoff, of the form

$$\mathcal{M}_{\text{theory}}\left(g_0^2, \ln\frac{\Lambda^2}{Q^2}\right) \tag{19.2}$$

The explicit Λ dependence goes away, if we replace $g_0 \to g(\Lambda)$ using the coupling defined by eq. (19.1). In perturbation theory, we would find that

$$\begin{aligned} \mathcal{M}_{\text{theory}} &= g_0^2 + g_0^4\left(\beta_0 \ln\frac{\Lambda}{Q} + c_1\right) + \dots \\ &= \alpha(Q^2) + c_2\alpha^2(Q^2) \end{aligned} \tag{19.3}$$

Now since $\mathcal{M}_{\text{theory}}\left(g^2(\Lambda), \ln\Lambda^2/Q^2\right)$ is independent of Λ, we are free to set $\Lambda^2 = Q^2$. Then the physical amplitude is

$$\mathcal{M}_{\text{phys}} = \mathcal{M}_{\text{theory}}(g^2(Q^2), 0) \tag{19.4}$$

where $g^2(Q^2)$ is *defined* by eq. (19.1), or explicitly

$$\frac{16\pi^2}{\beta_0 g^2(Q^2)} - \frac{\beta_1}{\beta_0^2}\ln\frac{16\pi^2}{\beta_0 g^2(Q^2)} = \ln\frac{Q^2}{\lambda_P^2} \tag{19.5}$$

and now the physical amplitude is

$$\mathcal{M}_{\text{phys}} = g_0^2(Q^2) + c_1 g_0^4(Q^2)\,(0 + c_2) + \dots \tag{19.6}$$

Comparing to (19.3), we can see that

$$\frac{g_0^2(Q^2)}{16\pi^2} = \alpha(Q^2) \tag{19.7}$$

since

$$\mathcal{M}_{\text{phys}} = \alpha(Q^2) + c_2\alpha^2(Q^2) + \dots \tag{19.8}$$

This makes clear what is the correct interpretation of the running coupling constant that I was criticizing in the previous lecture. The Q^2 dependence is not present in any fundamental coupling in the Lagrangian, but rather it arises from taking advantage of the Λ-independence of the physical amplitude and using our freedom to set Λ equal to the relevant scale of the process, to get rid of the log.[52] Of course, this is a consequence of solving the renormalization group equations, but this point of view seems to me simpler and more intuitive than the RG equations.

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[52] The preceding sentences are not in my notes, but seem to be the logical connection to the previous lecture.

19.1. Lattice QCD

Now we will move on to the main subject of this lecture, which is a comparison of different kinds of cutoff schemes, including the lattice and dimensional regularization.[53] As we have mentioned before, the path integral

$$\int e^{iS}\mathcal{D}A = \int e^{i\frac{1}{4g^2}\int F_{\mu\nu}F_{\mu\nu}d^4x+\dots}\mathcal{D}A \tag{19.9}$$

is meaningless until a UV cutoff is introduced. I find that the lattice is the most physically satisfying way of accomplishing this. We approximate spacetime as a lattice, and discretize all the field variables and notions of differentiation; for example

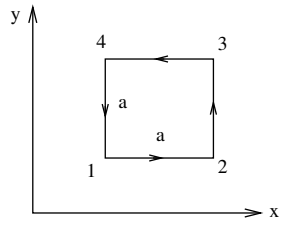
$$(\partial\phi)^2 \to \frac{1}{\epsilon^2}\big(\phi(x)-\phi(x+\epsilon)\big)^2 \tag{19.10}$$

K. Wilson invented this technique especially for solving QCD on the computer. In this framework, the gauge field A_μ is a connection that relates the relative orientations of the color frames at neighboring points on the lattice; these can be different from each other by an arbitrary SU(3) rotation. Hence it is natural to regard A_μ as living on the links connecting neighboring lattice points, rather than sitting on the lattice points themselves. Consider two neighboring points, labeled by 1 and 2, and define the link variable

$$U_{12} = e^{i\int_1^2 A^i_\mu \frac{\lambda^i}{2}dx_\mu} \tag{19.11}$$

where the path is a straight line connecting lattice points 1 and 2. The quarks, on the other hand, live on the sites. In this formulation, the U_{ij} become the dynamical variables rather than A^i.

Now we need to formulate the action in terms of the link variables. Of course it has to be gauge invariant. An invariant quantity must involve a closed path in configuration space,



$$: \quad U_{14}U_{43}U_{32}U_{21} \tag{19.12}$$

$$= \quad 1 + iF^i_{xy}\frac{\lambda^i}{2}a^2 + O(A^2)$$

The small $a\times a$ planar region together with this product is known as a plaquette. $U_{14}U_{43}U_{32}U_{21}$ is invariant under gauge transformations at any of the interior points $4, 3, 2$, but to make it also invariant at site 1, we must take the trace:

$$\mathrm{tr}\,[U_{14}U_{43}U_{32}U_{21}] = \mathrm{tr}\,1 + 0 + O(A^2) \tag{19.13}$$

The $O(A^2)$ term is the interesting part. Notice that

$$\begin{aligned} U &= 1 + i\sum b_i\frac{\lambda^i}{2} + \dots \\ \bar{U} &= 1 - i\sum b_i\frac{\lambda^i}{2} + \dots \\ \mathrm{tr}\,\bar{U}U &= \mathrm{tr}\,1 + \sum b_i b_i \tfrac{1}{2} + \dots \end{aligned} \tag{19.14}$$

But since U is unitary, it must be that all the terms of $O(b^2)$ cancel out, implying that we need to also keep track of such terms in the individual U matrices. It turns out that the relevant ones are

$$\begin{aligned} U &= 1 + i\sum b_i\frac{\lambda^i}{2} - \tfrac{1}{4}\sum b_i b_i + \dots \\ \bar{U} &= 1 - i\sum b_i\frac{\lambda^i}{2} - \tfrac{1}{4}\sum b_i b_i + \dots \end{aligned} \tag{19.15}$$

When we keep these second-order terms in the expansion of the U_{ij} variables in the plaquette, and expand the result to $O(A^2)$, the resulting expression is the finite-difference version of the free part of the gauge kinetic term,

$$F^a_{\mu\nu}F^a_{\mu\nu}\,a^4 \tag{19.16}$$

where $\mu\nu = xy$ in the example shown in (19.12). Summing on all plaquettes gives a sum on μ,ν as well as the sum over all locations, resulting in the action

$$S = \frac{2}{g^2}\sum_{\text{plaquettes}}\big(\mathrm{tr}[UUUU] - \mathrm{tr}\,1\big), \tag{19.17}$$

where the additive constant is unimportant. We must be careful about the relative orientations of the U's on the links, as indicated by the arrows in (19.12), to respect gauge invariance. This is not indicated explicitly in (19.17) but it should be kept in mind. The factor of a^4 in (19.16) represents the integration measure, since

$$\sum F^2\,a^4 \to \int d^4x\,F^2 \tag{19.18}$$

in the continuum limit.[54]

Next we turn to the interaction of gluons with quarks. How do we represent $\bar{\psi}\not{D}\psi$? Consider

$$\sum_{\text{links}}\bar{q}(x)\gamma_\mu\left[q(x+\mu)-q(x)\right] \tag{19.19}$$

[53] RPF called it "dimensional renormalization" in his lectures, but would have adopted the more common terminology for the version to be published.

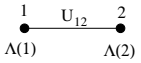
[54] I have in parentheses the question "take real part of tr[$UUUU$]?" in my notes. The answer is yes; for a nice review from this era, see J. Kogut, 10.1103/RevModPhys.51.659.

using the shorthand that $x + \mu = x + a\vec{e}_\mu$, in terms of a lattice unit vector $\vec{e}_\mu$ that points in the μ direction. But this is not yet gauge covariant, and it is missing the integration measure factor. Instead, take

$$ia^3 \sum_{\text{links}} \bar{q}(x)\gamma_\mu \left[U_{x,x+\mu} q(x+\mu) - q(x)\right] \qquad (19.20)$$

In fact, the second term $\bar{q}(x)\gamma_\mu q(x)$ can be dropped, since it has no effect on the dynamics, but just contributes an overall phase to the path integral.[55]

Let's look more closely at the gauge invariance of the gluon kinetic term. One can transform the color axes independently at each lattice site. Suppose we transform site 1 by the SU(3) matrix $\Lambda(1)$, and similarly at site 2 by $\Lambda(2)$. One finds that U_{21} changes by



$$U_{21} \to \Lambda^\dagger(2)\, U_{21}\, \Lambda(1) \qquad (19.21)$$

Therefore the action is invariant because

$$\text{tr}\left[\ldots U_{32} \underbrace{\Lambda(2)\bar{\Lambda}(2)}_{1} U_{21} \ldots\right] \;=\; \text{tr}\left[\ldots U_{32} U_{21} \ldots\right] \qquad (19.22)$$

This reasoning also shows how the quark kinetic term is invariant.

However we must still specify the form of the path integral measure for the link variables. Since U is an SU(3) transformation, a gauge invariant measure is required. This is known to mathematicians as the Haar measure. Since SU(3) is a compact group, $\int dU$ is finite at a given site, unlike the usual measure $\int dA$. We will not go into the mathematical details of the Haar measure here, but it is intuitively similar to the more familiar integration measure for rotations ($\sin\beta\, d\alpha\, d\beta\, d\gamma$ for SO(3) in terms of Euler angles).

Just like for a Pauli-Villars cutoff Λ, we need to find out how g must vary with the lattice spacing a in order for the theory to give results that are independent of a in the continuum limit, $a \to 0$. Technically, this is not so easy to do as for the Λ cutoff, but conceptually, it can be carried out in the same way. We would have to work out perturbation theory on the lattice to do it properly. Less rigorously, one can expect that $g(a)$ has a similar structure to our previous expression for $g(\Lambda)$ if we identify $\Lambda = c/a$ for some constant. Then the task becomes determining the correct value of c. Another way of thinking about it is in terms of the renormalization scale we called λ_P in eq. (16.1). We could define $\Lambda = 1/a$, but we would find that some other value of λ_P is needed to describe the same physics on the lattice relative to the Λ cutoff.

[55] I have elaborated here on what is written in my notes: "because it is just a number (?)"

To my mind, the lattice is the most concrete and least mysterious of all regulators, and it is the most physically sensible. But—and now I'm going to speculate—it seems like we are missing something by having to rely on these rather ad hoc schemes for defining our theories. It's comparable to Leibnitz and Newton inventing the integral calculus, which also looks like taking the continuum limit of a lattice,

$$\lim_{h\to 0} \sum_m f(x+mh)\, h = \int f(x)\, dx\,. \qquad (19.23)$$

The value of the integral does not depend on the machinery of cutting off the small scales and taking the limit, so one need not be preoccupied with the details of exactly how to discretize, like we are doing with all of our different regulator schemes for the path integral. We know how to do the ordinary integrals directly. Similarly we believe that the path integral has some kind of intrinsic meaning that does not depend on the cutoff scheme, but the difference is that we can't avoid that whole discussion, and the dependence on details of whether we use this kind of cutoff or that kind of cutoff. This makes me dream, or speculate, that maybe there is some way, and we are just missing it, of evaluating the path integral directly, without having to make this detour into the machinery of renormalization, since we know that the physics has to be independent of it in the end.

19.2. Dimensional regularization

Previously we already discussed dimensional regularization in a preliminary way. Here I would like to do it in somewhat more detail. The basic observation is that an integral like

$$\int \frac{d^D p}{(p^2 - m^2)\left((p-q)^2 - m^2\right)} \text{ converges for } D < 4\,. \qquad (19.24)$$

Hence we define $D = 4 - \epsilon$, and take the limit $\epsilon \to 0$ in the end. An integral like (19.24) gives a pole $1/\epsilon$. This idea was used to great advantage by K. Wilson to understand phase transitions in statistical mechanics, and it was applied to gauge theories by 't Hooft and Veltman:

K. Wilson, Phys. Rev. D7, 2911 (1973)

G. 't Hooft and M.J.G. Veltman, Nucl. Phys. B44, 189 (1972)

Exercise. Prove the following statements in D dimensions:
1. g^2 has dimensions of E^{4-D}.
2. Defining η_g and η_q as the number of external gluon or quark lines, and L as the number of loops, the power of g for any diagram is $2(L-1) + \eta_g + \eta_q$.
3. The dimension, in powers of mass, of the matrix element (amplitude) $|T|$ is $d = D + N - DN/2$ where $N = \eta_g + \eta_q$.

4. The dimension of the integral for a diagram is $d = 4 - \eta_g - \frac{3}{2}\eta_q - L(4-D)$.

Now I would like to consider the defining properties of the momentum space integrals in D dimensions, to show that they can be evaluated rigorously and with no ambiguity. There are four basic properties.

1. $\int f(p)\, d^D p$ is linear:

$$\int [\alpha f + \beta g]\, d^D p = \left(\alpha \int f + \beta \int g\right) d^D p \tag{19.25}$$

Among other things, this means that we can use Fourier and Laplace transforms in D dimensions.

2. Shifts of the integration variable are allowed,

$$\int f(p+a)\, d^D p = \int f(p)\, d^D p \tag{19.26}$$

where a is a constant vector.

3. Scaling:

$$\int f(\alpha p)\, d^D p = \frac{1}{\alpha^D} \int f(p)\, d^D p \tag{19.27}$$

4. Normalization:

$$\int e^{-p^2/2}\, d^D p = (2\pi)^{D/2} \tag{19.28}$$

since $\int du\, e^{-u^2/2} = \sqrt{2\pi}$.

We can define Fourier transforms in the usual way,

$$\begin{aligned} \int d^D p\, f(p)\, e^{2\pi i p\cdot x} &= \phi(x) \\ \int d^D p\, \phi(x)\, e^{-2\pi i p\cdot x} &= f(p) \end{aligned} \tag{19.29}$$

Next let us evaluate some integrals. Notice that

$$\int e^{-\frac{\alpha}{2}p^2} d^D p = \left(\sqrt{\frac{2\pi}{\alpha}}\right)^D \tag{19.30}$$

Then we can use the shift property to find that

$$\int e^{-\frac{\alpha}{2}p^2} e^{ip\cdot x} d^D p = \sqrt{\frac{2\pi}{\alpha}}^D\ e^{-x^2/2\alpha} \tag{19.31}$$

Other integrals can be generated from this one by differentiating with respect to α.

Exercise. Prove that

$$\int e^{-\frac{\alpha}{2}p^2} (a\cdot p)(b\cdot p)\, d^D p = X(a\cdot p) \tag{19.32}$$

and find X. Further prove that

$$\int e^{-\frac{\alpha}{2}p^2} p^2\, d^D p = DX \tag{19.33}$$

Another useful integral is

$$\begin{aligned} \int \frac{d^D p}{(2\pi)^D} \frac{(p^2)^R}{[p^2 - C]^M} &= \frac{i(-1)^{R+M}}{(4\pi)^{D/2} C^{R-M+D/2}} \\ &\times \frac{\Gamma(R+D/2)\,\Gamma(M-R-D/2)}{\Gamma(D/2)\,\Gamma(M)} \end{aligned} \tag{19.34}$$

The reader is invited to derive this one as well.

20. DIMENSIONAL REGULARIZATION, CONTINUED (1-21-88)

Another very useful class of integrals is that where the integrand f depends only on the magnitude of p. (Imagine that we have already Wick-rotated to Euclidean space.) Then

$$\int f(p)\, d^D p = C_D \int f(\rho) \rho^{D-1} d\rho \tag{20.1}$$

To determine C_D, we can consider the case where $f = e^{-\rho^2\alpha/2}$, since we already know the value of this integral from (19.30). Comparing with (20.1),

$$\begin{aligned} \sqrt{\frac{2\pi}{\alpha}}^D &= C_D \int_0^\infty e^{-\rho^2\alpha/2}\, \rho^{D-1}\, d\rho \\ &= C_D \int_0^\infty e^{-u\alpha/2}\, u^{\frac{D-1}{2}} \frac{du}{2\sqrt{u}} \\ &= C_D \left(\frac{2}{\alpha}\right)^{\frac{D}{2}} \underbrace{\tfrac{1}{2} \int_0^\infty e^{-u\alpha/2}\, u^{\frac{D}{2}-1}\, du}_{\Gamma(D/2)} \end{aligned} \tag{20.2}$$

hence

$$C_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \tag{20.3}$$

Exercise. Find a general formula for $\int d^D p\, f(p^2, a\cdot p)$. Hint: relate it to $\int e^{-\alpha p^2/2 + ip\cdot x}\, d^D p$.

To combine denominators, we can use formulas like[56]

$$\int_0^1 dx\, \frac{x^{m-1}(1-x)^{n-1}}{[ax + b(1-x)]^{m+n}} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \frac{1}{a^m b^n} \tag{20.4}$$

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[56] In my notes there is a question mark over the = sign and a parenthetical note to check the formula, probably a caution from RPF. It is correct.

Then

$$\int \frac{d^D p}{((p-q)^2-m^2)\,(p^2-m^2)} \tag{20.5}$$
$$= \int_0^1 dx \int d^D p \frac{1}{\left(p^2 - 2p\cdot qx + q^2 x - m^2\right)^2}$$
$$= \int_0^1 dx \int d^D p' \frac{1}{\left(p'^2 + q^2 x(1-x) - m^2\right)^2}$$
$$= \int_0^1 dx\, C_D \int_0^\infty d\rho\, \rho^{D-1} \frac{1}{\left(\rho^2 + q^2 x(1-x) - m^2\right)^2}$$

Claim:

$$\int d^D p\, f(p^2)(a\cdot p)(b\cdot p) = \frac{a\cdot b}{D}\int d^D p\, f(p^2)\, p^2 \tag{20.6}$$

Similarly,

$$\int f(p^2)(a_1\cdot p)(a_2\cdot p)\cdots(a_n\cdot p)\, d^D p \tag{20.7}$$
$$= \#\big[(a_1\cdot a_2)(a_3\cdot a_4)+\dots\big]\int d^D p\, f(p^2) p^n$$

where n is assumed to be even.

As we showed before, dimensional regularization reproduces the logarithms that we get from Pauli-Villars regularization, through the combination of the $1/\epsilon$ poles with terms like $(p^2/\mu^2)^\epsilon$. One shortcoming however is in the definition of chiral theories, since it is not clear how to define γ_5 in D dimensions, nor correspondingly the totally antisymmetric tensor $\epsilon_{\alpha\beta\gamma\delta}$. This of course is not a problem for QCD where parity is conserved.

A novel potential use of dimensional regularization, which is not usually considered, is that it could provide more than just a method for evaluating divergent loop integrals: it is also possible to use it to define a quantum field theory nonperturbatively in $4-\epsilon$ dimensions. I spent some time thinking about this, but was not able to get anything interesting out of it.[57]

20.1. Physics in D dimensions

Inspired by dimensional regularization, it is interesting to try to formulate a more complete picture of what physics would look like in an arbitrary number of dimensions. We start by imagining a linear vector space in D dimensions, with vectors x, y, *etc.* Given any two such vectors, linearity implies that

$$\alpha x + \beta y \text{ is also a vector} \tag{20.8}$$

We need a scalar product, a number associated with every pair of vectors,

$$x\cdot y \text{ maps } (\text{vectors})^2 \to \mathbb{R} \tag{20.9}$$

It must be linear,

$$(\alpha x + \beta x')\cdot y = \alpha\, x\cdot y + \beta\, x'\cdot y \tag{20.10}$$

and associative,

$$x\cdot y = y\cdot x \tag{20.11}$$

Now suppose we had four such vectors. We could construct the object

$$(a\cdot b)(c\cdot d) = a_\mu c_\nu b_\mu d_\nu \tag{20.12}$$

where the indices are no longer numbers taking on discrete values, but rather markers telling us which vectors should be dotted with each other. It is just an alternative notation. In this case, what would it mean to write an expression like

$$n_\mu = a_\mu c_\nu d_\nu \tag{20.13}$$

in which we have an index that is not contracted? Such an equation makes sense if we interpret it to mean that

$$x_\mu n_\mu = x_\mu a_\mu\, c_\nu d_\nu \text{ for all } x\,. \tag{20.14}$$

We can also think of n_μ as being a linear map from vectors into $\mathbb{R}$. And we can generalize this to several uncontracted indices, like

$$m_{\mu\nu} = a_\mu b_\nu + c_\nu d_\mu \tag{20.15}$$

which means that $m_{\mu\nu}$ maps pairs of vectors into $\mathbb{R}$, *etc.* Hence there is no need for the indices to take on discrete values as they would in an integer number of dimensions.

Contraction. In addition to operating on two vectors, $m_{\mu\nu}$ can be contracted on its own indices, $m_{\mu\mu}$, which is just a number in $\mathbb{R}$. For the above example, it is obviously $a\cdot b + c\cdot d$. However there is a special bilinear mapping,

$$\delta:\ (x,y)\to x\cdot y \tag{20.16}$$

that we call $\delta_{\mu\nu}$ in discrete dimensions. Then

$$\delta_{\mu\nu} y_\nu = y_\mu \tag{20.17}$$

is a consistent definition of $\delta_{\mu\nu}$. Now $\delta_{\mu\mu}$ is a pure number, that we are free to choose. Let us make the *definition*

$$\delta_{\mu\mu} \equiv D \tag{20.18}$$

Obviously, there is no restriction that D should be an integer.

Next let's consider how calculus should work. We can consider a nonlinear mapping, $F(x)$: vectors $\to \mathbb{R}$, such as

$$F(x) = \frac{x\cdot x}{(1+(a\cdot x)^2)^2} \tag{20.19}$$

[57] In the next lecture RPF expands on this, presenting a way to formulate quantum field without recourse to the path integral or canonical quantization.

What is the derivative of F? We define

$$D_c F \equiv \lim_{\epsilon\to 0} \frac{F(x+\epsilon c) - F(x)}{\epsilon} \tag{20.20}$$

where c is some vector. This is the directional derivative, along the c direction. It is a linear function of c, so it must be of the form $c\cdot$(something). Therefore we define

$$D_c F \equiv c\cdot\nabla F = c_\mu \nabla_\mu F \tag{20.21}$$

This specifies the ∇_μ operator, independently of c, since (20.21) must be true for any c. For the example (20.19),

$$c\cdot\nabla F = \frac{c\cdot\nabla\, x\cdot x}{\left[1+(a\cdot x)^2\right]^2} - \frac{2x\cdot x}{\left[1+(a\cdot x)^2\right]^3}\, 2a\cdot x (c\cdot\nabla)(a\cdot x) \tag{20.22}$$

as usual. Now

$$(c\cdot\nabla)(x\cdot x) = \lim_{\epsilon\to 0}\frac{(x+\epsilon c)\cdot(x+\epsilon c) - x\cdot x}{\epsilon} = 2c\cdot x, \tag{20.23}$$

so $\nabla_\mu(x\cdot x) = 2x_\mu$, as expected, and $(c\cdot\nabla)(a\cdot x) = a\cdot c$, giving $\nabla_\mu(a\cdot x) = a_\mu$ and $\nabla_\mu x_\nu = \delta_{\mu\nu}$.

Next we construct the Laplacian, by considering successive derivatives:

$$a_\mu b_\nu \nabla_\mu\nabla_\nu F = (a\cdot\nabla)(b\cdot\nabla)F \tag{20.24}$$

This allows us to isolate $\nabla_\mu\nabla_\nu F$, which is a tensor $T_{\mu\nu}$, whose contraction gives the Laplacian. For example,

$$\begin{aligned} \nabla_\mu\nabla_\mu\, x\cdot x &= 2\delta_{\mu\nu} \\ \nabla^2(x\cdot x) &= 2\,\delta_{\mu\mu} = 2D \end{aligned} \tag{20.25}$$

So far, all of our results look completely reminiscent of their counterparts in integer dimensions. But the concept of orthogonal subspaces leads to a novelty. Suppose we have a vector a such that $a\cdot a \neq 0$. For any vector x we can define the part that is orthogonal to a as

$$x' = x - \frac{a\cdot x}{a\cdot a}\, a \tag{20.26}$$

so that $a\cdot x' = 0$. Then any vector can be written as a piece proportional to a plus a piece in the orthogonal direction. Furthermore the dot product of two vectors splits into

$$x\cdot y = x'\cdot y' + \frac{a\cdot y\, a\cdot x}{a\cdot a} \tag{20.27}$$

If we take away the external vectors, this gives the definition of the Kronecker delta of lower dimensionality, living in the subspace orthogonal to a,

$$\delta_{\mu\nu} = \delta_{\mu'\nu'} + \frac{a_\mu a_\nu}{a\cdot a} \tag{20.28}$$

with $\delta_{\mu'\nu'}$ defined in the orthogonal subspace. Its trace is

$$\delta_{\mu'\mu'} = D - 1 \tag{20.29}$$

This procedure can be repeated to get subspaces of successively lower dimensionality, which of course terminates if D is an integer, but does not if D is noninteger. It means that we can construct infinitely many directions that are all mutually orthogonal in the noninteger case.

Exercise. Suppose that we have two vector spaces of dimension D_1 and D_2, and vectors x_1, y_1, x_2, y_2 respectively defined in the two spaces. There is a rule for combining them to make a vector of dimension D_3, such that

$$\begin{aligned} x_3 &= x_1 \oplus x_2, \quad y_3 = y_1 \oplus y_2 \implies \\ x_3 + y_3 &= (x_1+y_1)\oplus(x_2+y_2) \end{aligned} \tag{20.30}$$

Prove that $D_3 = D_1 + D_2$, and

$$\int F(p_3)\, d^{D_3}p_3 = \int F(p_1,p_2)\, d^{D_1}p_1\, d^{D_2}p_2 \tag{20.31}$$

21. PHYSICS IN D DIMENSIONS, CONCLUSION (1-26-88)

We have not yet discussed how to extend the Dirac algebra to arbitrary dimensions. Let's consider the gamma matrices. For any vector a_μ, we can associate a quantity

$$\not{a} = a_\mu\gamma_\mu \tag{21.1}$$

that has the property

$$\{\not{a},\not{b}\} = 2\, a\cdot b \tag{21.2}$$

Therefore $\{\gamma_\mu,\gamma_\nu\} = 2\,\delta_{\mu\nu}$ and $\not{a}^2 = a\cdot a$, where $\delta_{\mu\mu} = D$ as usual.

For the trace properties, I like to define two traces, that are normalized differently from each other. When acting on the unit matrix in the Dirac space they give

$$\mathrm{tr}[1] = 1; \qquad \mathrm{Tr}[1] = D\,, \tag{21.3}$$

so $\mathrm{tr}[X] = (1/D)\mathrm{Tr}[X]$. They satisfy the usual property

$$\mathrm{tr}[\not{A}\not{B}] = \mathrm{tr}[\not{B}\not{A}] \tag{21.4}$$

from which you can derive that $\mathrm{tr}[\not{a}\not{b}] = a\cdot b$ and $\mathrm{tr}[\not{a}] = 0$. In fact, by using the property

$$\mathrm{tr}[\not{a}\not{b}\not{c}\not{d}] = \frac{1}{2D}\left(\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] + \mathrm{Tr}[\not{d}\not{c}\not{b}\not{a}]\right) \tag{21.5}$$

and its generalization to an arbitrary number of gamma matrices in the products, one can demonstrate that the trace of any odd number of gamma matrices vanishes.

The Dirac equation in D dimensions can be written as

$$i\not{\nabla}\psi - V(x\cdot x)\gamma_0\psi = E\psi \tag{21.6}$$

for a spherically symmetric potential. It can be solved exactly; I have carried this out.

Exercise. Reduce the problem (21.6) to a conventional ordinary differential equation.

It turns out that the sum of two spaces of dimension 1/2 is not an ordinary 1-dimensional space. We would expect that

$$(x \cdot y)^2 = (x \cdot x)(y \cdot y) \text{ in 1 dimension}. \tag{21.7}$$

But this property does not hold when you construct it from two half-dimensional spaces, along the lines of the exercise at the end of the previous lecture.

Exercise. Show that a $D = 0$ space constructed from two spaces of equal and opposite dimension has nontrivial properties.

Newtonian mechanics is quite straightforward in D dimensions. We define a time-dependent vector $x(t)$ and impose the principal of least action, with the action

$$S = \int \left(\frac{m}{2} \left(\frac{dx}{dt} \cdot \frac{dx}{dt} \right) - V(x) \right) dt \tag{21.8}$$

Here I do not insist on a central potential; for instance $V(x)$ could have the form

$$V(x) = f(x \cdot x) + \sum a_i \cdot x \, g(x \cdot c) + \dots , \tag{21.9}$$

for example $V(x) = (a \cdot x + x \cdot x)/(1 + (x \cdot x)^2)$. You can prove that orbits stay in the same plane—this is just conservation of angular momentum—for a spherically symmetric potential; it's very dull.

One can solve the wave equation in noninteger dimensions,

$$\nabla^2 \phi = -\kappa^2 \phi; \quad \phi = \phi(x) \tag{21.10}$$

where $\nabla^2 = \mathbb{\nabla} \cdot \mathbb{\nabla}$, as we discussed in the last lecture. It can be done using the Fourier transform, which works in D dimensions, with $\phi = e^{ik \cdot x}$ as usual. If there is a source s, so that

$$\nabla^2 \phi = -\kappa^2 \phi + s \tag{21.11}$$

then by Fourier transforming one can solve

$$\tilde{\phi}(p) = \frac{\tilde{S}(p)}{p^2 - k^2} \tag{21.12}$$

and also do the inverse Fourier transform to obtain $\phi(x)$ in position space.

In quantum chromodynamics we would like to be able to generalize the path integral to D dimensions,

$$Z = \int e^{iS} \prod \mathcal{D}A(x), \quad S = \frac{1}{2g^2} \int E_{\mu\nu} \cdot E_{\mu\nu} \tag{21.13}$$

with $E_{\mu\nu} = \nabla_\mu A_\nu \dots$. Here $A_\nu(x)$ is just a vector field defined on the space of vectors, both in D dimensions. Therefore there is no difficulty in defining the action. However the measure is problematic because the spacetime is too abstract in D dimensions. In particular, we don't know how to construct a lattice if D is not an integer.

This motivates us to consider a different formulation of quantum field theory. Imagine some superfunctional that I will denote by $\{F[A]\}$, that acts on functionals $F[A]$ of the field in the same way as the usual normalized path integral, for the cases of integer dimensions where we know how to define it:

$$\frac{\int e^{iS} F[A(x)] \prod \mathcal{D}A}{\int e^{iS} \prod \mathcal{D}A} \equiv \{F[A]\} \tag{21.14}$$

Even though the definition of the left-hand side is not obvious for noninteger dimensions, we can generalize its known properties in integer dimensions to obtain a functional differential equation, that defines our mysterious superfunctional. Namely, we know that the path integral is invariant under a change of variables $A_\mu \to A_\mu + \epsilon \alpha_\mu(x)$, that I take to be infinitesimal. Moreover, the measure by itself is invariant under this trivial shift. Recall that the functional derivative $\delta F[A_\mu(x)]/\delta A_\mu(x)$ is defined by

$$\delta F = \int \epsilon \alpha_\mu(x) \frac{\delta F[A_\mu]}{\delta A_\mu(x)} \, d^D x \tag{21.15}$$

It follows that

$$\left\{ \frac{\delta F[A_\mu]}{\delta A_\mu(x)} \right\} + i \left\{ F \, \frac{\delta S}{\delta A_\mu(x)} \right\} = 0 \tag{21.16}$$

This equation is equivalent to the path integral, but more fundamental since it extends to the case of noninteger dimensions, and it can be used as our starting point. It is a statement of Schwinger's action principle, from which one can derive the usual perturbation expansion, but it can also serve as a nonperturbative definition of the theory.

One issue we have glossed over is the signature of the spacetime metric. This is a discrete choice in integer dimensions, and it is not perfectly clear how to deal with it in arbitrary dimensions. Should the fractional difference in the dimension be spacelike or timelike? In a more extreme case, what would the world look like if the metric of spacetime was

$$\delta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} ? \tag{21.17}$$

It might be advantageous to find some kind of geometrical description of events to answer this.

Although it is a somewhat different issue than noninteger dimensions, one could also think about derivatives of fractional order. We know that conventional derivatives have the form

$$\begin{aligned} \frac{df}{dx} &= \lim_{\epsilon \to 0} \frac{f(x) - f(x - \epsilon)}{\epsilon} \\ \frac{d^2 f}{dx^2} &= \lim_{\epsilon \to 0} \frac{f(x) - 2f(x - \epsilon) + f(x - 2\epsilon)}{\epsilon^2} \end{aligned} \tag{21.18}$$

What about a derivative of order 1.5 or 1/2? I will illustrate the correct generalization for the 1/2 order:

$$\frac{d^{1/2}f}{dx^{1/2}} = \lim_{\epsilon\to 0} \frac{f(x) - \frac{1}{2}f(x-\epsilon) + \frac{1}{8}f(x-2\epsilon) - \dots}{\epsilon^{1/2}} \tag{21.19}$$

where $-1/2$, $1/8$ ...are the binomial coefficients from expanding $(1+x)^{1/2}$. This turns out to be a valid procedure. And it has an inverse: you can find half-order integrals as well, in an analogous way.

Returning to dimensional regularization, I wanted to show in slightly more detail how the Q^2 dependence of the coupling comes out from the perturbation series in that method. Here I will set $D = 4+\epsilon$ and take $\epsilon \to 0$ at the end. Recall that in our Λ cutoff scheme, the form of an amplitude would look something like

$$\begin{aligned} g_\Lambda^2 \;+\; g_\Lambda^4 \left(\beta_0 \ln\frac{\Lambda^2}{Q^2} + c\right) + \dots \\ = \; \alpha(Q^2) + \alpha^2(Q^2)c + \dots \end{aligned} \tag{21.20}$$

In dimensional regularization, we must introduce an arbitrary mass m_0 to make the coupling dimensionless:

$$g^2 = g_0^2\, m_0^{-\epsilon} \equiv \alpha_0\, m_0^{-\epsilon} \tag{21.21}$$

The renormalization of the coupling constant now becomes

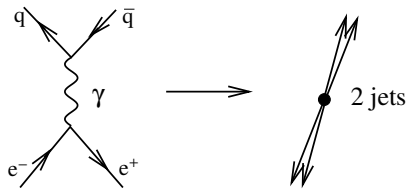
$$\begin{aligned} \frac{4\pi}{\beta_0\alpha(Q^2)} \;-\; \frac{\beta_1}{\beta_0}\ln\left(\frac{4\pi}{\beta_0\alpha(Q^2)}\right) \\ = \; \frac{4\pi}{\beta_0\alpha_0} + \frac{2}{\epsilon} + \ln\frac{Q^2}{m_0^2} + \gamma_E - \ln 4\pi \end{aligned} \tag{21.22}$$

where $\gamma_E = 0.5772\dots$ is Euler's constant. Nothing physical depends on m_0^2, nor on ϵ. By choosing the arbitrary normalization of m_0 appropriately, we can cancel the $2/\epsilon$ pole (the minimal subtraction MS scheme), or the pole along with the $\gamma_E - \ln 4\pi$ terms (modified minimal subtraction, $\overline{\rm MS}$ scheme).

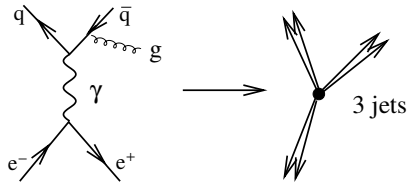
21.1. Scattering at high Q^2

We have seen that at high Q^2, the effective coupling of QCD is supposed to become small. But this by itself does not guarantee perturbation theory is necessarily very good, since the numerical coefficients of the expansion might turn out to be large. In particular, there is always a low-energy effect mixed in with any high-energy process because of hadronization of the final state particles. This part of the process is taking place at scales where the coupling is definitely not small and the calculation is not perturbative. It might seem like after all our wonderful efforts of using renormalization to improve the perturbative predictions at high energy, we could get foiled by these low-energy effects. However all is not lost, because we can separate these two phenomena from each other in a more or less clean manner. If we can argue that the details of hadronization are independent of the high energy scale Q^2, then this separation can be done quantitatively.

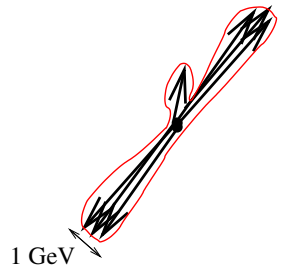
Recall that in e^+e^- collisions that produce $q\bar{q}$, we will see two hadronic jets,



or if a gluon is radiated at a large enough angle, we will see three jets,



At low Q^2, the third one is likely to be too soft to appear as a distinct jet. In this case the envelope of momentum vectors would appear to be a cylinder with a small bump for the third soft jet



At high Q^2, the cigar gets relatively narrower, and the jets become better defined. The gluon jet starts to become more distinct as Q^2 increases, like bringing a picture into focus. And as Q^2 continues to increase, greater numbers of jets start to appear.

We can measure the distributions of hadrons in jets at low Q^2, described by what are known as fragmentation functions. Although we don't know the fragmentation function for gluons, we can make some educated guess. The important point, related to my claim above, is that these functions do not change appreciably with Q^2, which leads to the factorization phenomenon that I described. These distributions take the form

$$c\,\frac{dp_z}{E}\, f(p_\perp)\, d^2p_\perp \tag{21.23}$$

where p_z is the momentum along the jet axis, and $p_\perp$ is the transverse momentum (shown as having a spread of $\sim 1\,$GeV in the picture above). To the extent that $f(p_\perp)$ is flat, within the jet, you will notice that this distribution is Lorentz invariant.

A useful quantity for characterizing particles in the jets is rapidity. Consider the quantity $E - p_z$, where $E = \sqrt{p_z^2 + p_\perp^2 + m^2}$ is the energy of a particle in the jet. Under a boost in the z direction, this becomes $\gamma(E + vp_z) - \gamma(p_z + vE) = \gamma(1-v)(E - p_z)$: it changes multiplicatively. Similarly $E + p_z \to \gamma(1+v)(E+p_z)$. Therefore the rapidity

$$w = \ln \frac{E + p_z}{E - p_z} \tag{21.24}$$

changes by

$$w \to w + \ln \frac{1+v}{1-v} \tag{21.25}$$

where v is the boost parameter. It is related to w by

$$v = \tanh w \tag{21.26}$$

and in terms of w a boost takes the form

$$\begin{aligned} E' &= E \cosh w = \frac{E}{\sqrt{1-v^2}}, \\ p' &= p \sinh w \,. \end{aligned} \tag{21.27}$$

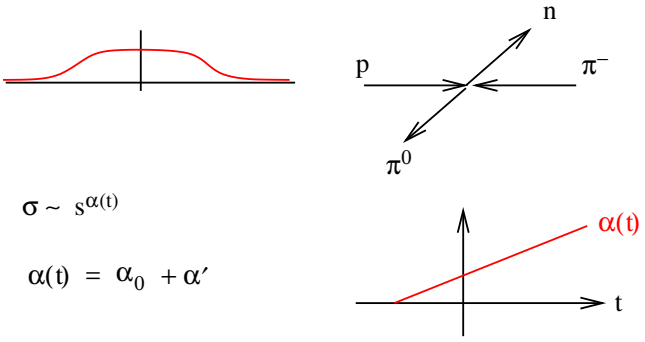
It turns out that the distribution $f(p_\perp)$ is not quite flat over its region of support, but instead goes as

$$\begin{aligned} f(p_\perp) &\sim \alpha(p_\perp^2)\, \frac{d^2 p_\perp}{p_\perp^2} \\ &\sim \frac{d \ln p_\perp^2}{\ln p_\perp^2} \\ &\sim d \ln \ln p_\perp^2 \,. \end{aligned} \tag{21.28}$$

Hence it is *nearly* constant, the log of a log. This small deviation from flatness has been observed and provides a confirmation of QCD.

21.2. Sphinxes

[The following figures appear without explanation in my notes.]



22. FINAL LECTURE (1-28-88)

22.1. Schwinger's formulation of QFT, continued

I would like to come back to the alternative formulation of quantum field theory that I started to discuss last time, eq. (21.16). To understand in more detail how to use it, let's consider the simpler example of a scalar field theory, where it takes the form

$$\left\{ \frac{\partial F}{\partial \phi} \right\} + i \left\{ F \frac{\partial S}{\partial \phi} \right\} = 0, \tag{22.1}$$

and now

$$S = \tfrac{1}{2} \int \left((\nabla \phi)^2 - \mu^2 \phi^2 - \lambda \phi^4 \right) d^4 x \,. \tag{22.2}$$

Remember that we are free to choose any functional F; it is instructive to take

$$F[\phi] = e^{i \int \sigma(x)\, \phi(x)\, d^4 x} \,. \tag{22.3}$$

From it, we can generate Green's functions by taking functional derivatives $\delta^n / \delta \sigma^n$. Then, with $n = 1$,

$$i\Sigma(x) \equiv \{ i\sigma(x)\, F \} = -i \left\{ F\, (-\nabla^2 - \mu^2) \phi \right\} \tag{22.4}$$

to zeroth order in λ. If we define $f(x) = \{ F\, \phi(x) \}$, we see that

$$i(\nabla^2 + \mu^2) f = i\Sigma(x), \tag{22.5}$$

which can be solved to get

$$f = \left(\nabla^2 + \mu^2 \right)^{-1} \Sigma(x) \tag{22.6}$$

This of course is just the propagator acting on Σ. Fourier transforming to momentum space, it reads

$$f(p) = -\frac{1}{p^2 - \mu^2}\, \Sigma(p) \tag{22.7}$$

To define what happens at the pole, we need to make an $i\epsilon$ prescription, as usual.

Now we can write

$$\{ \phi(x)\, F \} = \int dy \left\{ I(x-y)\, \sigma(y)\, e^{i \int \sigma\, \phi\, d^4 x} \right\} \tag{22.8}$$

where $I(x-y)$ is the propagator in position space. Hence

$$\left\{ \phi(x)\, e^{i \int \sigma\, \phi\, d^4 x} \right\} = \int dy\, I(x-y)\, \sigma(y) \, \left\{ e^{i \int \sigma\, \phi\, d^4 x} \right\} \tag{22.9}$$

or

$$\frac{\delta \{F\}}{\delta \sigma(x)} = \int I(x-y)\, \sigma(y) dy \, \{F\} \,. \tag{22.10}$$

We can integrate this to get

$$\{F\} = e^{\frac{1}{2} \int \sigma(x) I(x-y) \sigma(y)\, dx\, dy} \,. \tag{22.11}$$

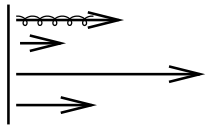
This is all at the level of free field theory so far. I leave it as an exercise for you to show that eq. (22.10) generalizes to

$$\frac{\delta\{F\}}{\delta\sigma(x)} = \int I(x-y)\,\sigma(y)dy\,\left[1-2\lambda\left(-i\frac{\delta}{\delta\sigma(y)}\right)^3\right]\{F\} \tag{22.12}$$

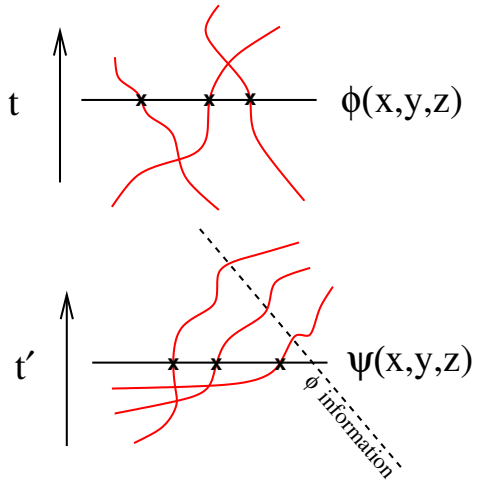
in the presence of the interaction. This can be solved perturbatively, or perhaps if you are clever enough, in some nonperturbative fashion. Obviously, there is no very sensitive dependence on the number of spacetime dimensions in this formulation, so it could serve as a nonperturbative definition of the theory in $4-\epsilon$ dimensions.

22.2. Parton model; hadronization

Now I would like to come back to some things that we started to discuss earlier in the course, during the first few lectures. Remember the parton picture of the proton,

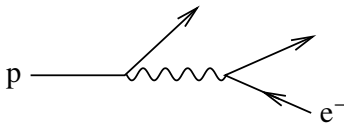


where the quarks and gluons have various momentum probability distributions inside the proton, that we denoted by $u(x)$, $d(x)$, *etc.* I had mentioned a conceptual problem, the fact that even if we knew the wavefunction of all the constituents for a proton at rest, this is not sufficient for determining $u(x)$, $d(x)$, ... The problem has to do with how the wave function transforms under a boost,

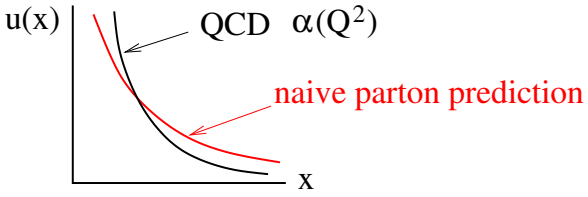


In one reference frame, at $t=0$, the wave function for three quarks at respective positions x,y,z is $\phi(x,y,z)$, while in some boosted frame, at $t'=0$, it is $\psi(x,y,z)$. It is a nontrivial task to get ψ from ϕ, since we need to solve the Schrödinger equation to propagate the quarks forward in time. As we saw before, this is complicated by the fact that the concept of the wavefunction is not relativistic.

But once we know the distribution functions at high energies, it turns out that they don't change very much as you go to even higher energies. We can determine these functions by doing proton-electron scattering,

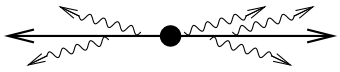


The reason that the distributions continue to change at higher energies is that the kinematics are not so simple as in this diagram: in reality, gluons are radiated, in particular in the forward direction, where we don't see them as distinct jets. In fact as we increase Q^2, gluons are more likely to be emitted, which leads to $u(x)$ depending on Q^2 and not just the momentum fraction x. This gives a correction to the naive parton picture, that neglected such effects. Qualitatively, the correction looks like



You might think that gluon emission should decrease at high Q^2 because of the running of $\alpha(Q^2)$, but it turns out that the increase in phase space outweighs this effect. It is similar to the infrared/brehmsstrahlung problem in QED.

Consider electroproduction of quarks, $e^+e^- \to q\bar{q}$. Many soft gluons will be emitted, but they all get lumped in with the quark jets, and are of little consequence *a priori*. Our concern is how this description may change as a function of Q^2.



Recall the fragmentation functions, $D^h(z,Q^2)$, that tell us the distribution of the momentum fraction z carried by hadron h in the jet. We would like to know how it depends on Q^2. Since the main dependence on Q^2 comes from the coupling, and this log dependence is also tracked by the cutoff, we can infer that

$$\begin{aligned}\frac{dD}{d\ln Q^2} &= \frac{\delta D}{\delta g_\Lambda^2}\frac{\partial g_\Lambda^2}{\partial \ln\Lambda^2} \\ &= \frac{\delta D}{\delta\alpha(Q^2)}\frac{\partial\alpha(Q^2)}{\partial\ln Q^2}\,,\end{aligned} \tag{22.13}$$

or, defining $\tau = \ln Q^2$,

$$\frac{dD}{d\tau} = \frac{\delta D}{\delta\alpha}\,\beta_0\alpha^2\,. \tag{22.14}$$

So we need $\delta D^h/\delta\alpha$, which arises at first order in perturbation theory, and is related to the probability for emitting a gluon (or possibly a q-$\bar{q}$ pair) that carries away some fraction $(1-y)$ of the momentum:

This leads to the evolution equation

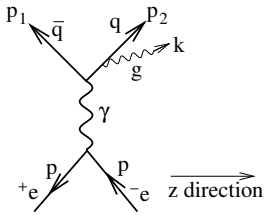
$$\frac{\delta D}{\delta\alpha} = \int \mathcal{P}(y)\, D^h(z/y, Q^2)\, \frac{dy}{y} \tag{22.15}$$

where $\mathcal{P}(y)$ is the probability of emitting a gluon that leaves the quark with momentum fraction y, if it was normalized to be 1 initially. One can show that

$$\mathcal{P}(y) = \frac{2}{3\pi}\, \ln\frac{Q^2}{m^2}\, \left(\frac{1+y}{1-y}\right) + \ldots \tag{22.16}$$

where the ... represent terms with weaker dependence on Q^2.

Exercise. Consider $e^+e^- \to q\bar{q}$ with the kinematics indicated below:



Show that the probability to emit the gluon is proportional to

$$\begin{aligned} P &\sim \alpha\, \frac{\epsilon_1^2 + \epsilon_2^2 + p_{1z}^2 + p_{2z}^2}{\epsilon_1\epsilon_2} \\ &\sim \tfrac{4}{3}\alpha(\epsilon_1^2 + \epsilon_2^2) \end{aligned} \tag{22.17}$$

where ϵ_i is the energy of quark i, and the second expression is the result from integrating over the quark directions.

We can write

$$\frac{dD}{d\alpha} = \tau\mathcal{P}\cdot D + \ldots \tag{22.18}$$

where $\mathcal{P}\cdot D$ denotes the convolution $\int \mathcal{P}D\, dy/y$, or defining $\kappa = \beta_0 \ln\tau$,

$$\frac{dD}{d\kappa} = \mathcal{P}\cdot D\,. \tag{22.19}$$

Hence if we evolve from Q_0^2 to Q_1^2, the change in κ is

$$\begin{aligned} \Delta\kappa &= \beta_0 \ln\frac{\tau_0}{\tau_1} \\ &= \beta_0\, \ln\left(\frac{\ln Q_0^2/\lambda^2}{\ln Q_1^2/\lambda^2}\right) \end{aligned} \tag{22.20}$$

This is a double logarithm, so the change is typically quite small. Take for example

$$\begin{aligned} Q_0 &= 6\,\text{GeV}, \quad Q_0^2 \cong 40\,\text{GeV}^2, \\ Q_1 &= 4000\,\text{GeV}, \quad Q_1^2 \cong 2\times 10^7\,\text{GeV}^2 \end{aligned} \tag{22.21}$$

This gives $\Delta\kappa \cong \frac{1}{2}$. To get a change as large as $\Delta\kappa = 1$, we would need to go to $Q_1 = 400,000\,\text{GeV}$!

So it is necessary to vary the energy quite dramatically to see any appreciable change in the fragmentation function D^h. And only by starting from rather low energies will we observe much variation at all. Notice that at such low energies as 6 GeV, many of our approximations that were appropriate for high Q^2 are not very good. The important point is that the details of hadronization are indeed insensitive to Q^2 as long as Q is well above the QCD scale.

Acknowledgment. JC thanks Isabelle Masse for proofreading and for helpful suggestions.

Appendices A-C are the verbatim transcriptions from the audio tapes of lectures 15, 17 and 18. Appendix D contains scans of RPFs hand-written addenda and corrections to revised drafts of two lectures. The remaining appendices are material that RPF handed out to the class participants, some of them written in his own hand.

Appendix A: Transcription: Scale dependence (1-5-88)

Okay. Now that we're starting the second term, we've formulated several times, in many different ways, in different kinds of gauges, the rules for perturbation theory; also the formulas in terms of path integrals. One of the purposes of path integrals is a statement of the equations which is not strictly speaking necessarily simply perturbation theory. If there was some way to compute the path integrals, and there is for instance numerically, there would be a scheme for making calculations which would not rely simply upon the need for perturbations. Because of our limited ability in doing path integrals until the present time, we're only pretty good in perturbation theory. We got so good at it from working with quantum electrodynamics where the coupling constant is very small, and therefore we've had lots of practice. But please don't think that we have to do everything by perturbation theory.

During this term we are going to talk in the first half of it about perturbation theory and what we can learn of quantum chromodynamics[58] *from perturbation theory. The second half of the term will be an attempt to understand the behavior of this theory, the fact that it confines*

[58] RPF says electrodynamics but it is clear he meant to say QCD.

quarks and so on, in some way by looking at the path integrals, without actually expanding them in perturbation theory. It will not be mathematically accurate; it will be qualitative. Say this will get big, this will get small, I think this will be bigger than that, and therefore this will happen. You will be very dissatisfied if you want precision. One of our problems as we're discovering, right, is doing these things with precision. The state we are in now is one where we will have to discuss it in a qualitative way. So that's all I can do, but we'll do that in the second part of the term. In the first part of the term, we will find out how much we can do by perturbation theory to test the theory.

So far we made only passing reference to the running of the coupling constant. This subject requires some care to avoid confusion. But I must say that it would not be at all difficult, there would not be any particular problem, and it is a very simple matter; the confusion comes because we can't calculate anything, and so we try to say as much as we can without calculating . . . It's something like the subject of thermodynamics, which appears to be quite complicated, but if you use always the same variables, such as temperature and volume, to represent the system, it's much simpler than if you suddenly say now wait, let's suppose I want to plot this on entropy and pressure; it's the perpetual change of variables from one to the other that makes the subject so complicated.

So there is a certain apparent complexity here, which is due to our inability, or indefiniteness, in choosing a method of calculation, or indefiniteness in deciding what process to calculate. That makes it look a little complicated. Let me make believe, at first, that we could calculate whatever we want. Then the problem would be the following. It would be straightforward. We would start with our theory, with Lagrangian

$$\frac{1}{g_0^2}FF + \bar{\psi}(i\partial\!\!\!/ + m_0)\psi \ldots$$

Let me put the constant and call it g_0. And then in the part that has to do with the quarks, with different flavors, there would be masses for the different flavors. So there would be a number of constants which are in the theory,

$$g_0^2,\ m_0^u,\ m_0^d,\ m_0^s,\ m_0^c,\ m_0^b$$

I'm putting a subscript 0 on them, which means those are the values that we put into the equations; in case there's any question of what I mean, that's what I mean: the numbers that we put into the original Lagrangian to make the calculation. So this is perfectly definite. There's the c quark; we may discover one day that there are others [quark flavors] *so we might need a few other parameters, but for the future.*[59] *So at the present time we've got these six numbers, the parameters that we can put into the theory.*

Now suppose we start out with this theory, we put some parameters in, and we compute something: the mass of the proton, the mass of the pion, and so on. If we computed six quantities, and we could compute perfectly and the theory were right and experiments were available for all those quantities, then we could determine these parameters. Then if I computed a seventh quantity, that would be a completely predicted quantity and we begin to test the theory. That's simple and straightforward and that's all there is to it, **except** *. . . First of all, we can't compute the mass of the proton, so we can't determine these constants. In view of that you also would know that it would be very practical to compute some quantities rather than others.*

For instance, if you wanted to compute the mass of the b quark, you could probably do a pretty good job by trying to compute the mass of the Upsilon, which would be pretty close to the mass of two b's, $2m_0^b$. We've got a very crude beginning for m_0^b without being able to calculate. We would get m_0^b most likely from masses of the Upsilon and its excited states, and we learn about the interaction strength somewhere, we correct for the excitation states that would make a very accurate value of $2m_0^b$, which would presumably be very close to one half of the mass of the Upsilon—reasonably close to a half the mass of the Upsilon, in other words to, uh, 5, 10 GeV [speaking to himself] *the mass of the Upsilon is . . . so it has to be 5* [for m_0^b]*. Yes that's right. In the same way the mass of the c could be determined as being about 1.8. Now, if we did the calculation more elaborately, you see we would be picking out the light quantities; instead of having the mass of the proton and the mass of the pion looking for some tiny deviation which was due to the c quarks, which are hardly affecting either one of them, neither one of them, then to get the* [c quark] *mass, that's not the way to do it.*

So it's sensible to try to pick out physical quantities that are more sensitive to particular parameters than to others. Of course in principle you could compute any old six quantities with infinite accuracy and deduce all the parameters. But it would be more practical to choose six quantities that are more directly sensitive to the masses. I've already got rid of m_0^b . . . .

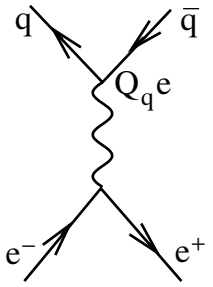
Now we want to try to concentrate on things which depend—because there's a great deal of interest in that quantity—physical quantities that depend upon g_0^2, and are not very sensitive to the others. I believe that one is the mass of the proton, because the mass of the u and the d . . . we have good evidence are very small, and that the mass of the proton is not due to the mass of the quarks inside, I mean at least not directly . . . it has to do with the value of g_0. It's hardly sensitive to m_0^s, but we could imagine that someday we could correct for that . . . However at the present time it is hopeless to compute the mass of the proton from these constants theoretically, and therefore we can't determine g_0 from the mass of the

[59] The top quark had not yet been discovered at this time, though its existence was not doubted.

proton, even though someday we could.

Another kind of effects where I look for g_0 are those phenomena that use high energy and which—so high that the masses of these things [quarks] *don't make any difference. And insofar as these* [quark masses] *are involved, we can presumably compute their effect. In other words we look for processes in which we expect there would be a limit, that limit would still exist for this process if the masses of these things went to zero.*

I'll give you some examples. This is only to suggest things to look at that we can calculate that will help us to isolate parameters, in particular g_0. One interesting experiment is $e^+e^- \to$ hadrons.



(A.1)

And the idea of that is if you do that at high energies, the electron and positron annihilate and produce a photon, which you can understand. And then the photon produces a pair of quarks, as a sort of initial disturbance. We have an operator $Q_f \bar{\psi}_f \gamma_\mu \psi_f$ for each quark flavor, and this operator starts by generating a pair of quarks. Now what happens after that is of course that this [quark] *maybe radiates a gluon, the gluon splits into quarks, and they combine together and they make π's and they make K's and you get a big splash of junk. The total cross section for doing this is the chance that we got started, so to speak. You can calculate the probability that we got these things started by just figuring that they're free, because the energy is so high. And then after a while they scratch their heads and say, "hey, I'm not supposed to be able to come out, I've gotta do something else," but they're already there. I don't know if you feel this intuition very well, I get it myself but I don't know how to express it; that at high enough energies, when we start this process and then after that ...*

Let's put it this way: suppose you did this, and you thought one day that you made a ρ. The next time you realize that you're not going to see the ρ but the ρ actually disintegrates into a pair of pions. Well the fact is, by the time you got to the ρ, and then it went into π's, it isn't going to change the total cross section; whether the ρ does or doesn't disintegrate doesn't make any difference to the total rate. The ultimate things that happen to these objects are not much affected by things that happen late, and therefore at low energy, and therefore involving the masses of the quarks and so on. So therefore the total cross section shouldn't involve the masses of the quarks. It shouldn't involve anything in QCD, there's no coupling constant at all [the QCD coupling] *because this rate to produce this pair of quarks that act like free particles, we calculate it directly.*

In the same way we could compute the rate to produce a pair of μ's and calculate that pure electrodynamic thing and call it $\sigma_{\mu\mu}$. Then I could calculate the probability of producing hadrons here [in the diagram (A.1) with hadronization of the quarks]*; the cross section would be the cross section for producing μ's—which also doesn't depend much on the mass of μ—we take a very high q^2 so the masses don't make a difference; then we would have the cross section for making, let's say, u quarks. Then the charge is $2/3$ for a u quark and the cross section goes as*

$$\sigma_{u\bar{u}} = 3\left(\frac{2}{3}\right)^2 \sigma_{\mu^+\mu^-}$$

to produce u quarks; so this would be the probability of producing u quarks. We can produce the u quarks in three colors: red, green and blue, and therefore the thing is multiplied by 3. Is there any question about that? Yeah? [a question is asked, inaudible] *No sir, because this is an electromagnetic phenomenon I'm talking about, this is a* [photon] *not a gluon, so this coupling is not QCD. Any other questions? You had me for a minute* [laughter].

This **three***, I say you can produce any color but that's a lot of nonsense, you can't produce any color, because you have to go into a singlet state. So you're going to go into the state*

$$\frac{1}{\sqrt{3}}\left(R\bar{R} + B\bar{B} + G\bar{G}\right) = \frac{3\,a}{\sqrt{3}} = \sqrt{3}\,a \qquad \text{(A.2)}$$

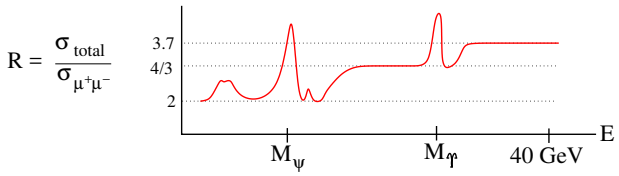
Now let's say the amplitude to go into the red-anti-red state, what I first calculated over there, let's call that amplitude a. Then the amplitude to go into blue-blue would also be a, and the amplitude to go into green-green would also be a, so the amplitude to go into this state would be $3a$ times $1/\sqrt{3}$—this is a normalized state—which is $\sqrt{3}a$. And the rate is $3\,a^2$. In other words, three times the rate of making $R\bar{R}$. You can fake it if you like, i.e. sloppily, that there's the same chance for red quarks or green quarks or blue quarks, therefore I add them—multiply by three. Or realize that you don't produce that state at all but you produce a superposition. But you come out with the same answer.

So that's for u quarks. But then we might produce d quarks also. And by the same method of thinking,

$$\sigma_{d\bar{d}} = 3\left(-\frac{1}{3}\right)^2 \sigma_{\mu^+\mu^-}, \ \textit{etc.}$$

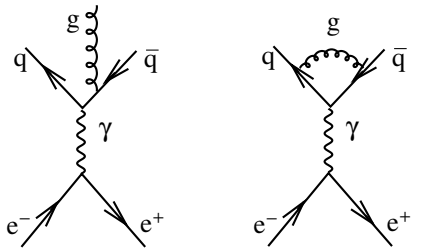
And now, we might produce s quarks. Now s quarks are not very heavy, so if an e^+-e^- experiment is done at several GeV, or 10 GeV or something like that, then that's another one third [writes on board] *... And then, if we have enough energy to get above the c, we produce c quarks* [writes on board] *and then maybe the b; depends, but if we have a total energy between say 9 and 10, probably 9.65 ... Then there's another factor yet; for the ratio*

of the total cross section to the cross section of the μ, we get a curve



which is an interesting thing, it bobbles around, there are bumps and things for the ρ meson . . . And as you get to high energies, here's 3.6, it comes here and there's a wonderful resonance . . . and makes the ψ and then some particles, and then the background comes up here, and over here it starts to make the Υ . . . until we have 30 or 40 GeV, and we don't see the resonance we expected for the t quarks. But it does show more or less constant as long as you're in a region where you're not at the same order as a mass of a new kind of quark that you can make. And it rises from these various plateaus with these numbers [gotten by adding the squared charges from the previous calculations] . . .

At any rate, that's **marvelous**, *but that doesn't determine* **any** *of those constants—too bad. However, maybe the idea if we do this at high energy, and worry ourselves about the b quarks and the c quarks and the interactions, we might be able to get a little accuracy this way. There is, of course, an interaction; the trick is . . . that it doesn't involve the masses.*



(A.3)

. . . and therefore that should be a good approximation.

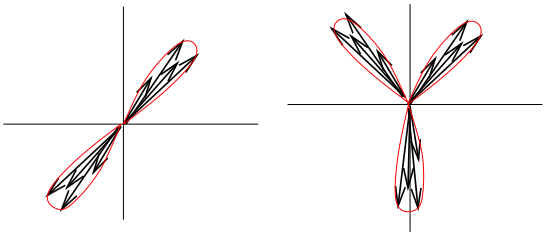
On the other hand there is the possibility that we can calculate what happens with the possible emission of a gluon . . . It could be that emitted a gluon. Or it could be that there were interaction forces between these [quarks] *by the exchange of a gluon. Now it is not quite as obvious that the effects of these things will not depend on the masses of the particles; but calculations by putting masses in show that it really doesn't. And that there is a correction that now involves, as you would like, the coupling constant g which we discussed. And what happens is, that we get that the same theoretical ratio* [as we discussed before] *is multiplied by a correction,*

$$R = R_{\substack{\text{pure free}\\ \text{particle}}} \left(1 + \frac{\alpha_g}{\pi} + \dots\right) \tag{A.4}$$

which is proportional to the coupling constant. I'm going to use α_g[60] *. . . strong interactions, put the g just to remind you for gluons . . . plus higher terms. This, then, would be a way, if we could measure accurately enough, to determine the α_g, and therefore g_0. It would be most sensitive to g_0.* [Question from me: what was that subscript you put on R, those words?] *The words say, "pure free particle" theory. This is the real ratio, corrected to the first degree for quantum chromodynamics. You can make a power series expansion in the coupling constant, the first term of which is α_g/π. And this then is a way to determine a quantity which is particularly sensitive to g_0^2 and which is presumably not sensitive to the other masses, although we do have to do a little work to get rid of these things, we make corrections for these things, depending on what region of the graph you want, we correct for that mass . . . and we can do a fairly good job of correcting for the masses . . .*

So that's one way of getting a quantity which depends on g and it would be a possible thing. The trouble with it is it's a correction to an experiment which gives 99— [pauses to think and correct himself, in undertone] *no this is I think this is 5% of the total—95% of the answer, that doesn't depend on quantum electrodynamics* [QCD] *at all, and you've only got a 5% correction, it's not very easy, you can't do it very well. So it would have been a nice thing, and it would have been a nice experiment, so if we're talking* **ideally** *that would have been a place to look for something to calculate and to measure to get something that's practically dependent on g_0.*

Alright, now there's another thing that's observed. According to this model, here, when we knock these two quarks out, and they're going very fast, and then they just tear out and I don't know what, radiate gluons, and do all kinds of things, they fall apart and make whatever strings there are . . . and what happens is that if we look at the **momenta**—



—I draw it in a plane because it's three-dimensional, the momenta, what we get is **thousands** *of hadrons, lots and lots, most of them pions. Okay. And if we plotted the momenta, and I'm only going to plot it in two dimensions instead of three, we find that they're all distributed in sort of a—at least if q^2 is very large—in a kind of a long ellipse, which is much longer than it is wide;*

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[60] Noise interferes, but I believe he says that $\alpha_g = g^2$, with the usual factor of 4π absorbed into his unconventional definition of g.

this is . . . the order of, well if you added all the momenta . . . conservation of energy, it's a rather big number . . . but they have a certain width, the width is of order of a half a GeV . . . Now, suppose you try to calculate the chance that this happens, and that this thing goes off . . . you remember what I do, I have to calculate this correction to this diagram [RPF is apparently explaining how the three-jet configuration on the right arises from radiation of a gluon from one of the quarks as in the previous diagram (A.3)] *and I add that to the rate of this* [the 2 jet].

Now in this case I could look at this one [the 2-jet] *and one would notice that there's a small chance that what happens, looking down on the plane, is that geometrically there's a momentum like that; you've got one quark coming out over here in this direction, and this quark starts out this way, if you want, if I employ the virtual diagram. What you see, though, is a quark coming this way, and a gluon going that way, the total momentum of which balances this* [the other jet]. *So if we didn't see this region here, it would seem . . .*

That can happen, at wide angle, you can ask for wide-angle gluons coming out. What we see experimentally is that from time to time, it doesn't look like this [presumably the 2-jet diagram], *it looks more like this* [the 3-jet diagram]. *And if this is sufficiently . . .* [probably referring to the hardness of the gluon jet] *and stuck out . . . then we can interpret it as being this* [the 3-jet picture]. *Why I have to say that is of course, if it's not sufficiently obvious, it might be that* [the 2-jet picture] *with a fluctuation. I mean if these two are close enough together, how can you tell the difference? You can't. So you'd have to take the case where there's a pretty good angle, which turns out to be a low chance. It's low because, you see, when these two open out . . .* [RPF explains on the board that in this case where the gluon is hard, the extra intermediate quark propagator is carrying large momentum, which suppresses the amplitude]. *So this happens rarely, but we can see it. And although it does, from the phenomenological point of view does involve soft masses and so on in determining whether we get π's or K's and how many, we can at least count how many jets we get, and estimate these momenta fairly well.*

The only uncertainty is whether we should include a particular particle here; is that part of this jet or part of that jet? So there is some sloppiness in it. The sloppiness will become less as the energy of the experiment is increased. There's some sloppiness, but we can do a pretty good job of guessing that these things come through this [2-jets], *and by measuring their rate, we get a pretty good idea of this* [3-jet] *rate, that has an α_g directly, because the amplitude for this process has a coupling constant in it, and this rate has a coupling constant squared, so this is a direct measurement of α_g . . .*

Later on in the course therefore we're going to calculate that . . . and compare it to experiment to try to determine the coupling constant. Okay? I've therefore illustrated two examples, and there are others, . . . that seem to isolate experimental data and seem to be able to measure, that we can roughly calculate. I say "seem to be able to measure" because we have all these little uncertainties, so it's nothing exact, but a pretty good measurement and a pretty good calculation by which we can determine α_g today and prove our claims . . . So that is a perturbative effect of something that depends mostly on α_g.

But what I wanted to explain mainly, the main thing I wanted to explain, is that there **are** *physical processes, for which we can say that we don't need to know the masses of the other particles. That the process has the same limiting value, the same probability, whether these are all zero or not. It's not so easy to correct for them; it depends on the energy. Let us, since we've demonstrated more or less that there are such processes, let us assume that there are some physical data, which involve just that* [g_0^2] *and not these* [the quark masses], *because we make corrections for this, and there are physical phenomena for which we believe, theoretically, the phenomena will still exist as the masses went to zero, of the quarks, and it doesn't involve any other length* [scale].

So we could imagine those things, those are special kinds of data that don't involve the masses of the quarks, and without saying so, from now on I'm talking about that kind of data. When I talk about a physical quantity, I'm going to suppose it's that kind of a quantity, okay? Not something like the mass of the π, which depends on the mass of the u, or the mass of the K, or the difference between the Λ and the proton, which is certainly dependent on the mass of the s. Those are not the kind of thing that I want to talk about here. Alright? . . . This will focus our attention.

Now we're ready to go, huh? No! Another complication sets in to . . . The theory doesn't make sense. The perturbation theory gives infinities . . . the theory diverges. You all know that. I don't have to prove it to you; we'll discuss it all later in detail. We're going to go back over all this and do it. I'm describing where we're going to go and what we're going to see. So the theory has infinities. In the case of electrodynamics, which you studied, and as you know what this does in that case is that there is some kind scheme for cutting off all the integrals that are divergent, in other words

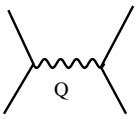
$$\int \frac{d^4k}{k^2(p-k)^2}\,.$$

Then this is divergent logarithmically because you've got four k's down here and four k's up there. And what people do is they say that the propagator for instance for a photon is replaced by subtracting from it what you would get if the propagator had a mass and then taking the limit as the mass goes to infinity,

$$\frac{1}{p^2} \to \lim_{\Lambda\to\infty} \frac{1}{p^2} - \frac{1}{p^2-\Lambda^2} \tag{A.5}$$

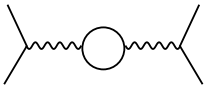
and obtain results which have logarithms in them; you have cutoffs, actually, and what do you do with these infinities? What we do with these infinities is the following.

If we discuss for example the scattering of two electrons by a photon at large distances, where the potential is e^2/r, or the scattering amplitude is



$$\sim \frac{4\pi e^2}{Q^2} \tag{A.6}$$

If we were to compute this, at very low Q, and long distances, then this [the coupling e] *is an experimental number. And now we discuss that this experimental number is not the same as the number e_0 that I would put into the theory, right away, because there are virtual diagrams*



where this may make a pair and the pair may annihilate, and like that, and this has got that kind of a divergence in it, and corrects this [(A.6)]. *What we've discovered is that* [taking his time to write it from memory]

$$\frac{1}{e^2_{\rm exp}} = \frac{1}{e_0^2} + \frac{1}{3\pi}\ln\frac{\Lambda^2}{Q^2}$$

Yes. Now what we're supposed to do is to make Λ go to infinity and this gets some kind of nonnegative result. Q is the momentum transfer, which is supposed to be small. But what we say is, when we do the theory with a cutoff, we change the theory, because the theory by itself doesn't mean anything. So we use a particular cutoff, and we take an e_0 such that the physics is independent—is correct, agrees with experiment. We choose a cutoff Λ and an e_0 so that it agrees with experiment. And now if we change the cutoff, we change the e_0 so that it continues to agree with experiment. In this case, we could change the e_0—if we change the Λ we could change the e_0 so that this ...

It's very important, the right sign, and I get the right sign by [long pause while RPF checks that the sign of the running is correct]. *That's 137* $[1/e^2_{\rm exp}]$.[61] *As Λ gets very big, this* [the log] *can get bigger than 137, so this* $[1/e_0^2]$ *would have to go negative which makes no sense. This theory really doesn't work, because it means this would have to go negative; I must be missing a sign... In any case, to encounter this problem you need to put in such a large cutoff that the logarithm is $3\pi \times 137$ which is more than 1000. Then we would need $\Lambda \sim e^{500}\, Q$, which is a mass greater than the mass of the universe. So there is no practical problem. So that's why we repeat all these calculations, without ever getting into any trouble, in practice. In practice, we never have to take the Λ so large to get a good accuracy ... From a theoretical standpoint that seems satisfactory.*

Alright. So what the trick is, and the point is, is that when we're putting our cutoff down, we're changing the theory. And when we use different kinds of cutoffs, we're using different theories. However, as it turns out that by putting different g_0's in, we can get the same physics from the different theories. For each theory you have to have its g_0, for each Λ there has to be an e_0 ...

But what happens in electrodynamics [QCD] *is better, because the sign is the other way around, and this is what happens. We could ask, for any experimental data ... we could ask the following thing. ... So we first have to modify the theory, to make it work. Really, we have to define a process of calculation. There are several methods. One is the method I mentioned before* [(A.5)],

1. $$\frac{1}{k^2} \to \frac{1}{k^2} - \frac{1}{k^2-\Lambda^2}$$

plus some tricks to keep gauge invariance. It can be done, it's not very good about that without those tricks for gauge invariance. ... [Tape was changed here. RPF is describing dimensional regularization.]

2. *dimensional regularization.*
   α *becomes dimensionful when $N \neq 4$;*
   $\alpha = \alpha_s(\Lambda_D)^{4-N}$

... as I will show you, physics with a fractional number of dimensions. So we calculate with dimension N. And then we write $N \neq 4$. As it turns out, when the number of dimensions is N, the coupling constant, which I'm going to write as α instead of g^2, the coupling constant α has a dependence on a scale, an energy scale; that is, it's not a [dimensionsless] *number, it's not a pure number as it is in four dimensions; it has dimensions. So if I put a dimension, say Λ_D or something which is some kind of a length, to a certain power, namely $N-4$, then this thing in front* $[\alpha_s]$ *will be a constant, as we will vary Λ ...*

Third method: we replace space and time—spacetime—by a lattice of points in spacetime. And to define what to do on that lattice, which is analogous to the Lagrangian here, and I will discuss that, that's called the lattice model.

3. *Lattice model.*
   Has a dimensionful parameter:
   $$a = \frac{1}{\Lambda_a}$$

The lattice model has a dimension to the lattice, how small it is. The dimension a corresponds to an energy—I'm going to talk about energies—$1/\Lambda_a$; this is the spacing of the lattice, the lattice spacing. Later on in the course, I will discuss both of these methods, not so much this one, but I'll talk about this one too.

Oh by the way! This method is not defined yet, we also have to say in what gauge we do it in. There's an

[61] Recall that RPF prefers to normalize the gauge couplings as $\alpha = 1/e^2$.

axial gauge, there's a $\partial_\mu A_\mu = 0$ gauge, which propagator, whatnot. They're all variations on a theme. All I'm trying to say is, all these things are mutilations of our beautiful scheme. Mutilations which we have to make, because otherwise it's all meaningless. However, we expect the following. Let's take the lattice model; it's the easiest to understand. Surely, if the number of points taken is sufficiently fine, we'll get a damn good representation ... [he compares to numerical algorithms for approximating differential equations as finite-difference equations] *if we don't get enough accuracy we make the lattice smaller. For any particular size of lattice, there are artifacts—errors—if you want to talk about something this big and the lattice is that big ... The size of the physical phenomenon here would be something involving the reciprocal of the momentum cubed ... Then the q* [momentum] *is the reciprocal of the wave number, and we would like to get our constant* [a] *smaller than the wave number, or we'll never be able to represent that ... But in order to get good accuracy, if we want to make it really good, we should make it still smaller, because as we all know, the more fine we have the lattice, the better the representation. This is some sort of limit, that if we took the lattice fine enough, we should get more and more perfect agreement with everything.*

However ... so we would like these Λ's—or in this case it's a question of how fast you come to the limit $N = 4$, and in this case it's a question, surely putting something like that [the wrong-sign propagator in Pauli-Villars regularization] *is going to change the physics if Λ^2 is not enormous compare to Q^2. Right? The propagator is different—you changed it. But you don't think you changed it much. So, we have the idea, and it turns out to be right, that if we have a phenomenon at a certain scale Q, and if we take Λ's much bigger than Q, and we consider different Λ's, we can always find a constant* [bare coupling] *to put in that gives the same physics. At first sight you might not have realized that you'll have to change the g_0. You would have thought that's going to be fixed, but the theory with its divergences shows that in the same way as in electricity there are going to be logs ...*

And so what happens is, when we ask ourselves, how to choose—let's pick a datum to analyze, any one of them ... We imagine we could calculate it. Then we ask how to choose the g_0—the other constants are involved, but we suppose that we select the datum as something sensitive to g_0 ... [RPF is illustrating this on the board, but I do not have it copied in my notes] *... so the physics is always the same. Or the physical datum comes out always the same. The datum agrees with the theory .... With one datum we can always make it agree with the theory by adjusting for a given Λ^2 by picking out the g_0^2. For another datum, we might not get perfect agreement, if the Λ^2 wasn't large enough, right? Because the theory is a little dopey at low Λ^2. If it was a lattice, the lattice scale is too big. So we really want this in the limit of very, very large Λ. We aren't interested in the formula for a Λ of the same order as Q or $6Q$ or something like that. That's not the problem, because things will work for one datum but the other one won't work, because the theory isn't right on the scale of ... Is there any question about this idea? I've tried to explain the idea, and I hope that if you don't catch on tell me what's bothering you, and we'll straighten it out now.*

[Question from a student, inaudible] *That's right, we're assuming that this thing is going to work. Yes, this business of choosing a lattice is analogous to the usual one, choosing a lattice for, say, doing the diffusion equation. And you expect it to become more and more accurate as you make the lattice finer. The only complication is that we have to keep changing the g_0 as we do so, in order to keep the physics the same. But the presumption is that we* **will** *be able to do that, that there will be a definite limit. That's an assumption which is, well I don't know whether people can claim they have proved it, but it seems to be true. Okay?*

Anyway we want the physics to be the same when we change the Λ^2, but also, by the way, when we change the method [of regularization]. *Like I said* [referring to a previous illustration on board], *various Λ^2's and various methods; I've labeled the Λ's differently for different methods, but it's the same idea. And I'm now going to tell you the answer ... and later on I'll prove it ... Is there another question? Okay, the answer is that the value of g_0^2 that you have to choose has the following expansion in* $\ln\Lambda$*:*

$$\begin{aligned}\frac{1}{g_0^2} &= \beta_0 \ln\Lambda^2 + \frac{\beta_1}{\beta_0}\ln\left(\ln\Lambda^2\right) \\ &\quad + c + \left(\frac{a}{\ln\Lambda^2} + \dots\right) \end{aligned} \tag{A.7}$$

where the further terms that are small [falling with $\ln\Lambda$]*; these don't interest us. We're supposing that Λ is big enough ... β_0 and β_1 are computable and known. c is arbitrary; it's where we have the room to choose, to make the thing fit the data. That's the constant we choose to make it fit the data, so c is arbitrary, you can have any constant there, and it's chosen to fit the data. But the formula, and what you have to choose, depends on the method—by the method I mean whether you use the lattice method or dimensional renormalization or what method of ultraviolet ... you use. a* [in (A.7)] *also depends on the method; β_1 and β_0 do not depend on the cutoff method.*

[Question about the arguments of the logs not being dimensionless.] *Yes, now that's ridiculous, isn't it? Very good, Λ's are energies. And I've written logs that aren't any—ha ha ha ... Well, the old professor can fix that, we'll divide by $(1\,\mathrm{GeV})^2$:*

$$\begin{aligned}\frac{1}{g_0^2} &= \beta_0 \ln\frac{\Lambda^2}{\mathrm{GeV}^2} + \frac{\beta_1}{\beta_0}\ln\left(\ln\frac{\Lambda^2}{\mathrm{GeV}^2}\right) \\ &\quad + c + \left(\frac{a}{\ln\Lambda^2} + \dots\right) \end{aligned} \tag{A.8}$$

But why $1\,\mathrm{GeV}^2$? We'd better contemplate that some

other joe will come along and use $10\,\mathrm{GeV}^2$ *in there. So we'd better contemplate what happens if instead of writing with* $1\,\mathrm{GeV}^2$ *underneath with* M^2 *underneath, before somebody else comes along, he likes to write his thing different. He would write this. And I would like to explain to you why it doesn't make any difference. Because that makes it look that it is still more arbitrary, that we have another thing . . . we don't. And I'll show you why. We might have a different constant* $[c']$*, that's the clue:*

$$\begin{aligned}\frac{1}{g_0^2} &= \beta_0 \ln\frac{\Lambda^2}{M^2} + \frac{\beta_1}{\beta_0}\ln\left(\ln\frac{\Lambda^2}{M^2}\right) + c' + \frac{a'}{\ln\frac{\Lambda^2}{M^2}} + \dots \\ &= \beta_0 \ln\Lambda^2 + \frac{\beta_1}{\beta_0}\ln\underbrace{\left(\ln\Lambda^2 - \ln M^2\right)} + (c' - \beta_0 \ln M^2)\dots \\ &\qquad\qquad \ln\Lambda^2\left(1 - \frac{\ln M^2}{\ln\Lambda^2}\right) \\ &= \beta_0 \ln\Lambda^2 + \frac{\beta_1}{\beta_0}\ln\ln\Lambda^2 + \left(c' - \beta_0 \ln M^2\right) \\ &+ \left(a' - \frac{\beta_1}{\beta_0}\right)\frac{1}{\ln\Lambda^2} + \dots \end{aligned} \tag{A.9}$$

You see that all I did was change the constant. If I had used "1" in here [for M]*, I've got a certain constant, when I made them fit the data. If I had used 10 or* M*, I'd get a different constant, that's all. So I'm still okay, right? The* M *makes it look as if there is another parameter, which adds to the confusion of this damn thing. Because when you hear about choosing* M *and choosing* Λ *and choosing* g_0*, there's no choosing* M*, really: it doesn't make any difference. It's just a question of the definition of the constant when you go to fit the data.*

Now the part that I hadn't finished is here; you notice that as Λ^2 *goes to infinity . . .* [RPF explains that the regularization-dependent term a' becomes negligible as the cutoff is removed.] *So I must take my* Λ^2 *big enough that this term doesn't amount to anything. So everything's okay, and that answers your question about the units. Is that alright?*

Sir? [Question from me: Is this equation just the two-loop approximation?] *Yes—no, it's exact!* [me again: there aren't more logs of logs?] *Oh yeah, maybe down here, there's log log, this times log log, stuff like that. But always smaller, okay? No, there's no log log log, no. There's no log log log. I'll explain to you why. It may be wrong in that there may be a term—I'm not sure, okay, like* $\ln\Lambda$ *times* $\ln\ln\Lambda$*, or something like that, which is still smaller than this one, but not much. And things like that, but these are all dropping out as* Λ *goes to infinity. So, I should say, terms of this order or smaller are going to drop out, that's what this curly line means. Okay? All we have to do is to take the* Λ^2 *very large and then we can do that.*

Alright. And that's the formula how then we believe, that if we do that, choose the g_0^2 *so it's equal to this and adjust the constant c, we can fit the first physical datum. And then a second physical datum . . . what with the same thing, with the same constant, you should get a fit to experiment, provided that we've chosen the* Λ^2 *large enough that this* [the a term] *is small enough that everything's okay. Alright. Are there any questions? That's all there is to it. That's all there is to what we call the way the theory's supposed to work.*

Review. The theory diverges. In electrodynamics that's really serious [the Landau pole problem] *but we don't pay any attention to it for practical reasons,* e^2 *is sufficiently small . . . In quantum chromodynamics because of the opposite sign, there isn't any real difficulty* [because of asymptotic freedom]*. We* **can** *choose the g in terms of the method that we use to make the cutoff in such a way that we would expect that the physical data agree with experiment. That is what the theory is. The theory is, strictly speaking, not the Lagrangian which we wrote down or the path integral we wrote down, but the path integral plus all this crap about how to make a cutoff, plus this baloney about how we have to choose g. And then we should take the limit as* Λ *goes to infinity to get the most accurate result.* **That's** *the theory. The theory of quantum chromodynamics is* **not** *defined by the Lagrangian alone. To put it another way, you cannot say to a mathematician "hey, here's my Lagrangian . . . , figure out the consequences," because you haven't told him the full physics of what you intend to do, which is—because if you give him that he'll find out that the answer is infinity—it doesn't make any sense. The true theory is the Lagrangian plus a cutoff scheme, plus a proposition as to how the g's go, so that the results will be independent of* Λ*, the cutoff scale, as the cutoff scale gets sufficiently fine. And this has to be the way to do it.*

Now you can find this out by perturbation theory, of course. You'll notice that as Λ *gets very large, the* g_0^2*'s are very small. When the g's are small we can compute everything by perturbation theory, and that's the way you computed the* β_0 *and the* β_1*. Well why didn't we compute c? Because you can't; it's an arbitrary constant. And why didn't we compute a? Because that depends on the method. We can compute it for each method, but it doesn't do us any good. I'll tell you why. This is a signal that there will be errors of* $1/\Lambda^2$ $[1/\ln\Lambda^2]$ *in the end, because we haven't taken a fine enough lattice spacing, using the example of the lattice. Because the lattice doesn't really represent the continuum. And this is a kind of measurement error. So it's no use to compute this "a" accurately for a given scheme . . .*

Yes? [Question: Λ is something you choose arbitrarily?] *No, we try to make it as large as possible.* [student: We try to make it as large as possible, but we can **choose** it to be as large as possible.] *Yes, that's true.* [student: Okay, it seems like we can always choose Λ large enough to make g small enough that we can use perturbation theory] *That's correct* [student: and then we can use perturbation theory on any problem.] *That's correct.* [student, not satisfied: I have been told that . . .] *But the series diverges . . . for processes with small Q, small momentum transfer, the corrections to the propagator get bigger and bigger* [student: even when g is small?] *What hap-*

pens is, when you take this exchange between two quarks ... you get corrections ... say a loop of gluons or something, which modify the propagator between two quarks. Now this thing, when you calculate it, involves something like $\ln \Lambda^2/Q^2$*, when you calculate it. This interaction will now will have a term like this with an extra* g_0^2*. Although the* g_0^2 *is small, the* $\ln \Lambda^2$ *is undoing it, and you get a finite ... which isn't small. It gets to be small in its effect if* Q^2 *were big enough, as we will learn next time, but if you ask the question at low* Q^2*, it just doesn't work. The divergences of the perturbation theory undo the smallness of the ...* [bare coupling in the UV]. *Okay?*

Alright. Well the difficulty with the perturbation theory is not that it doesn't exist; it's that you can't sum it. We don't know how. Sometimes we can sum some terms, but we can't do a very good job, we have to think about it, rather than calculate it, even though with sufficiently large Q^2*, the effect of* g *is small, for smaller* Q^2 *the effects are bigger ...*

[The following apparently refers to eq. (A.8) or (A.9).] *I'll be putting a mass squared here from time to time, maybe ... you'll appreciate that it doesn't represent an independent choice of ... It does—there is a way of making it look like an independent choice. Obviously, there is ... suppose that we finally fix the data and worked it out and determined c. And somebody could find an M so that this canceled out. And then he could say that this formula for* $1/g_0^2$ *is exactly this. And there's no constant and the other constant is M. And there's all these different ways of representing the same thing, which causes a tremendous amount of confusion to a lot of people, and I'm sorry for that, because we really ... calculate it, so we don't know the M; some people come out with* $0.2\,\mathrm{MeV}$ *or something like that*[62] *and say that's what it is,* $0.5\,\mathrm{MeV}$*, whatever. So they don't know it well enough, so we don't have the numbers accurate enough that we can do one or another of these things once and for all and be done with it, so we have to kind of leave all these balls in the air, as to which way you would prefer to write it—whether to choose an M and say that it's the constant* [c] *I want to determine, or to say the constant I'm going to choose is zero and it's the M I want to determine. So you'll hear different people saying different things, but you have to understand that they're all equivalent. It always makes it a little easy to do it on the blackboard because I've prepared the lecture, but then you have stop and think are they really equivalent ... or you'll forget how I did that.*

Are there any other questions about the idea? As you can probably see, because of the logarithms, you might expect $1/g_0^2$ *is not really converging, and if we are working at* $100\,\mathrm{GeV}$ *or something like that, and you wanted to change* Λ *to change the logarithm in order to get something that ... it's damned hard! ... And therefore something like the lattice ... They have a technical thing; they can do some calculations on lattices ... The lattices really aren't small enough to get a good answer ... They're as small as they can make them and still do the calculation, because of the number of ... that are available ... But when they try to make the* Λ[63] *smaller to be more accurate, they need a lot more computer time, because in four dimensions if you decrease the lattice by one half, you have sixteen times as many points to compute. Sixteen times as much work. But even changing the lattice by* $1/2$ *is only changing* Λ *by a factor of two, and* $\ln \Lambda$ *doesn't do much. And so it is very difficult, in fact I would say virtually impossible, to make the numerical calculation practical ... limitations of computers. To make the numerical calculations with greater accuracy ... So I think we have to study this theory, not only to figure out an analytic way ... to understand well enough what happens at short distances ... so that we have a better way of computing that is less sensitive to this brute force scheme that they're now using. During this course I will discuss all the numerical calculations, and more of these methods, and everything else; this is just an introductory lecture to explain where we have to go.*

Appendix B: Transcription: Renormalization: applications (1-12-88)

[A student is asking whether RPF is going to explain the correspondence between dimensional regularization and the cutoff to which he has been referring so far.] *Yes, I am. I worked it out the other day and it's very simple; I understood it* [RPF says something to the effect that he might get some details wrong here since he is going by memory] *The point is that in dimensional analysis* [regularization] *the coupling constant has a dimension, so you represent the physical coupling constant as g times some dimension—some energy, which corresponds to our* Λ*—to a power of 4 minus the number of dimensions. Now when you do an integral over correction terms in perturbation theory, all of those integrals converge if the number of dimensions is less than 4, and so the corrections to the coupling constant—that's what I'm trying to get straight ... Now as the d approaches 4 it turns out—let's say we write* $d = 4 - \epsilon$*—then there's an* ϵ *down here as you approach ... It ends up that you're trying to work out something like* $(1/\epsilon) \times \Lambda^{-\epsilon}$ *which gives you* $1/\epsilon$ *times* $1 - \epsilon \ln \Lambda$ *... I was trying to get that straight just before I came ... I couldn't figure it fast enough ... but there's a direct correspondence ...*

We have been talking about the renormalization of the coupling constant in a kind of abstract way, and as usual at the beginning of each lecture I have to fix up some minor things ... in order to make everything I said conso-

[62] RPF meant to say $0.2\,\mathrm{GeV}$, referring to the scale Λ_{QCD} that he usually denotes as λ_P.

[63] RPF means the lattice spacing

nant with the outside world, all the equations in which I wrote g^2, in all those the g^2 should be replaced by $g^2/16\pi^2$; then

$$\alpha = \frac{g^2}{4\pi} \tag{B.1}$$

This is the real world. This is me.[64] *My α was πg^2 times 4π, which is not good to do. This is the right thing; first you substitute this, then you substitute that . . . All the equations are changed with the appropriate positions of the 4π's. There was one equation that we chose to define $\alpha(Q^2)$, and that was $1/\alpha(Q^2)$, which now becomes*

$$\frac{4\pi}{\alpha(Q^2)} = \beta_0 \ln \frac{Q^2}{\lambda_P^2} + \frac{\beta_1}{\beta_0} \ln \frac{4\pi}{\beta_0\, \alpha(Q^2)} \tag{B.2}$$

The equations I'd written before didn't have the 4π's. Alright?

Now I will just remind you of what we discovered, that when we did perturbation theory to order g_0^2 to some process, that's all the order we worked out. And this is replaced ultimately by $\alpha(Q^2)$ . . . that's in perturbation theory; it becomes this if you would sum the leading logs, because you always know they come in common. It can't be g, because that's $\ln \ln \Lambda$ *. . . nothing depends on the cutoff . . . In other words, when you do first order perturbation theory, you simply replace the g_0's by $\alpha(Q)$, and you get a much more accurate result. You've included all the higher order leading logs. If you try to do it to next order in the coupling constant, it gets a little more complicated. You get*

$$g_0^2 + g_0^4 \left(\ln \frac{\Lambda^2}{Q^2} + a \right) \tag{B.3}$$

—I'm not going to try to get my 4π's right—and we know already there was one of these [in the argument of the ln] *in there, that has to be there, and that's just to get the right coefficient. And then there will be some constant* [a] *that has to be worked out when you do the second order perturbation. If that's the case, then this turns into*

$$\alpha(Q) + a\, \alpha^2(Q) \tag{B.4}$$

So the way to do second order perturbation theory, is after you do it, take away the logarithmic term and just look at the constant term, and the constant term is the coefficient of the second order term in α. And it will be a little more complicated with the 3rd order term. But we can work it all out. And it tells you, in other words, from the perturbation expansion . . . to write it not in terms of g_0^2 but in terms of $\alpha(Q)$. It's the effect of summing the leading logs, that are evidently going to come in, although we haven't worked it out for the higher terms. This is a much harder . . .

So in fact therefore people say that the coupling constant is dependent on Q^2 because of the running—that's what running means . . . I just want to say it again, it looks complicated but it's relatively simple; at first order you replace g . . . at second order . . . .

There's many papers and places where you can read about this process of renormalization; I mentioned a [version of ?] *the renormalization group equations which will look much simpler than any ones you'll see anywhere, unless of course I have it physically right and have done it nicely. The problem is making sure that the quantity that we're dealing with depends . . . And that the quantity we're dealing with is physical and doesn't involve something like just a Green's function, an expectation of ϕ at one point and ϕ at another point. Because the ϕ's–wave functions, or field operators—also shift their coefficients with various c's and so on. If you deal with a physical quantity that you measure, you don't have any of that stuff. For example, if we have an expectation of vector potentials*

$$\langle A(x_1) \cdots A(x_n) \rangle$$

at one point times another point or something like that, then you have to watch out that these vector potentials are also changing their definition as we change the coupling constant.[65] *So in the renormalization group equations for Green's functions . . . well it looks much more complicated, but it really isn't that much more complicated, it's just the physical ideas were adequately described . . . it's always a good idea to stick to physical questions . . . So a lot of . . . book or any paper on renormalization, you find it enormously more complicated than anything you've seen; they've added all kinds of extra stuff. It also has a lot about the history where people tried this and that and did this and did that and proved it this way and proved it that way . . . The subject looks worse than it is. Okay. So try to look it up and . . . that I cheated you somehow in describing it . . . I did find a nice book called* Renormalization *by John Collins, Cambridge University Press, 1984. I can't read it all, it's too complicated for me. Now there's* **thousands** *of references on renormalization . . .*

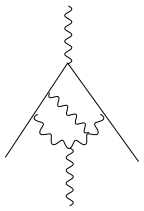
Now that's the end of last time—I'm always fixing up the lecture before. Oh, there's one more [thing to fix up]*; I've forgot the β_1/β_0 here* [eq. (B.2)]*, and I'd better tell you where the π's are . . . The fact that this is positive, if it was only about electrodynamics and didn't have the solution to write the conclusion to be negative; it also suggests, the fact that it's positive means that everything will work . . . wonderful theory . . .* [RPF refers to

[64] Apparently RPF made a side-by-side comparison of the two notations, but I only copied the "real world" version in my notes.

[65] In my notes I have written that RPF is talking about anomalous dimensions here.

the Landau pole problem of QED] *However if the number of flavors is more than something like 17, then we're in trouble. So most people believe there is not 17 flavors of quarks. We only know of five so far. It might be six, since people like to fill out the symmetry with t quarks. But there is no theoretical reason to say that there isn't another group of three, like u, d, s—(u,d), (s,c)–strange and charm, and you have beauty and truth or something . . .* [inaudible, some other word starting with "t," drawing laughter from the class]*. And maybe there's x and y. And maybe there's w and v—we don't know; there's no understanding as to why there's more than one family, or why it stops at three families . . .*

Okay, well, . . . the problem is how do we actually calculate something . . . and perhaps this whole problem . . . and you'll see it all coming out . . . So that's what I'm about to do. But even there, before I do a particular problem, I want to do something about guessing where the divergences are going to come. We all know from doing quantum electrodynamics and other field . . . I'm assuming you've taken a course in field theory . . . comes out to be divergent because of . . . And so let's try to find out when we're going to get . . . So if you were to take a very complicated diagram to calculate, some crazy thing . . . there's a quark, quark, gluon, gluon . . . it doesn't make any difference . . . something like that . . .



$$\int \frac{d^4k\, d^4p}{(p^2,\ k^2+\cdots)^6} \times f(p,\ k, \cdots) \quad \text{(B.5)}$$

And then such an object will end up with two integrals over momenta, one for this loop and one from this loop, four-dimensional integrals . . . Therefore we need to integrate over eight variables, and in the end the question is does it diverge? The real question is, will there be—what kind of formula are we going to integrate? There will be various k^2's, minus this and that, propagators, maybe there'll be six propagators. I don't mean there's a sixth power of the propagator, I mean there are six of these kind of things in a row. Perhaps . . . So I'm not going to worry about whether it's p or k. But there will be from the gradients in the couplings, up here [the numerator f] *there will be some k's and p's. And then the question is when we go to do these integrals, we will get a divergence, a logarithmic divergence or . . . depending on how many powers are down here and how many powers are up here. If there's more powers downstairs than there are upstairs, then it will be a convergent integral.*

So what we have to do is count how many powers there are upstairs and downstairs. And that means looking at all these couplings and seeing if there are gradients in them, taking two powers for every propagator of gluons, one power for every propagator of quarks . . . Now I think there are no gradients in the coupling of a gluon to a quark and so on. And we get all . . . and have a big counting job. And it will depend on the structure of the diagram. And now, for a miracle. There are very many relationships between the way diagrams are constructed, and what kinds of topology you can take. Of course one of the typical theorems of topology is that . . . I don't know why that should be relevant for this because it doesn't have to be a planar diagram, that [66]

$$Edges + Vertices = Faces + 2 \quad \text{(B.6)}$$

In other words, there's a relationship between the number of loops, the number of vertices and the number of propagators . . . But what I'm trying to say is that the number of loops, the number of junctions, the number of couplings, and all this stuff are not completely independent of each other, but they're related to each other. And those relationships turn out to mean that I can make this count and tell you the answer in a very nice way, it's very simple. The net power of any integral, the number of numerator over the denominator—in this case for example, one, two, four and four is eight . . . there are twelve down here, let's say there's one more here, that would be fourteen, so that's a net of minus two, so that's convergent.[67] *The net power of all the momenta, which is N, has this property:*

$$\begin{aligned} &N = 0, \ \textit{log divergence} \\ &N = 2, \ \textit{quadratic divergence} \\ &N < 0, \ \textit{converges} \end{aligned} \quad \text{(B.7)}$$

and the wonderful thing is, no matter how complicated the diagram, the formula is that

$$N = 4 - \mathcal{N}_g - \frac{3}{2}\mathcal{N}_q, \quad \text{(B.8)}$$

four minus the number of gluon lines coming in from the outside, minus 3/2 the number of quark lines coming from the outside, period! It doesn't make any difference how it's all structured in there. That's an entertaining thing; you can play around and try to prove it to yourself . . . by actually counting things up and showing various relations of the number of intersections and junctions and three-point couplings. See, for example, this relation between the number of junctions and the number of lines, because each line has two ends, so you know, take the number of lines divided by two, it's going to tell you how many junctions there are. At any rate, this ends up as being true, which is most remarkable. Well now I'm going to prove . . .

[Question from student: . . . superficial degree of divergence?] *Yes, yes, yes, superficial degree of divergence. It*

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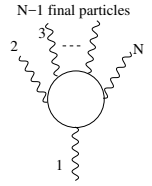
[66] In my notes I have written that this assumes a vacuum diagram, no external legs.

[67] The answer for the diagram in (B.4) should be -1

is often called a naive counting divergence because, what could happen, is that this whole problem, turns out there are momenta up here, but they're not the momenta of the integrand that you have to integrate over, but they might be the momenta of the outside lines—let's call q the typical momenta coming in, that we're not integrating over. Then dimensionally, from the point of view of the number of energy terms, it's the same dimension, but the integration is more convergent. [Student: Or what could happen is that we have two integrals, over k and q; the q integral is very convergent, and the k integral is divergent.] *That could happen, but it usually doesn't.* [Another student: but anyway this is a worst case, this counting?] *Yes.*

Let me explain how I did this. A way of looking at it, one way of a direct count, the most obvious way . . . Here's another way. This, we could say, is part, it's a diagram for some process. It's a piece of a lot of terms that are going to be added together to produce a matrix element T for a process. Now since they're going to be added together, they all have the same dimension, so the dimension of this is the same as the dimension of T. But in the case of T, we have various rate formulas. Let's take an example. We have a single particle going into . . . I don't care how, and disintegrating into $N-1$ particles, one coming in and $N-1$ coming out. Then we say that the rate at which this happens goes like this:



Rate of decays of 1 into $N-1$ particles:

$$d\Gamma = \frac{1}{2E_1}|T|^2 \prod_{i=2}^{N}(2\pi)\delta(p_i^2 - m_i^2)d^4p_i \times (2\pi)^4\delta^{(4)}\left(p_1 - \sum_{i=2}^{N} p_i\right) \qquad \text{(B.9)}$$

including the (2π)'s—which have nothing to do with dimensions, but I'm being accurate for a change—that's very rare.[68] *That's the exact formula for the rate. There's a similar formula for the cross section, but that's with two particles coming in; let's just take the case of one.*

What I'm going to use it for is to determine the dimensions of T. By the way, in this case it's very important that the coupling constant has no dimensions. We're going to have coupling constants, g, g, g, g all over this thing, and those g, g, g, g's is not going to make any difference to the dimension. So all I have to do is find the energy dimension of this thing, and thereby obtain the maximum degree to which it could possibly diverge. Okay, now the rate is one over the lifetime of that object, and which is therefore an energy,

$$[Rate] = \left[\frac{1}{\tau}\right] = [energy] \qquad \text{(B.10)}$$

[68] The factors of $(2\pi)^{-4}$ for each d^4p_i are missing

and multiplied by this energy [moving E_1 to the left-hand side]*, we get*

$$[energy^2] = [|T|^2]\,[\text{energy}]^{2(N-1)}\,[energy]^{-4}$$

Okay? And so we find out that

$$\begin{aligned}[|T|^2] &= [E]^{8-2N} \text{ or} \\ [|T|] &= [E]^{4-N}\end{aligned} \qquad \text{(B.11)}$$

where N is the total number of lines coming out. And that's what this formula [(B.8)] *was supposed to be, only it's slightly . . . Because we then can prove—you have to watch out, we talked about the dimension of the integral that we're going to get when we do this. Well that's not quite the same, because whenever a quark comes in, there's a spinor for that quark, and that spinor has a dimension. So the integral* [the loops inside the diagram, with external wave functions removed] *is not the same as the dimension of T. But you have* [the dimension of] *the integral is four minus the number of gluons minus the number of quarks, which is what I have there* [(B.11)]*, that's T, but for each of the quarks there was a spinor, which has a dimension of* $1/2$*, and so the integral doesn't quite have the same dimension as the T,*

$$I = 4 - \mathcal{N}_g - \mathcal{N}_q - \tfrac{1}{2}\mathcal{N}_q \qquad \text{(B.12)}$$

To remind you, that a quark or spinor has a dimension of $1/2$*, you remember that if you're going to sum this over spins, when you sum this over spins and there are quarks in it, what you do is you say, oh I know, I'll get some kind of a matrix element, and then I put a projector* $\sum u\bar{u} = \not{p} + m$ *in it when I sum over spins. That means when you sum over spins, you put an extra energy in. So the dimensions of the T when summed over all the spins has these factors, one of these for every quark. That was* $|T|^2$*, therefore half an energy for each quark, so that's where this* [last term of (B.12)] *comes from.*

And now finally if you have some good reason to know that the final answer . . . for some reason you know, gauge invariance might be such a reason, that you know the final answer must have zero divergence [in terms of contracting external momenta with the amplitude]*, for a gluon with momentum q that it has to come in this way,*

$$q^2\,\delta_{\mu\nu} - q_\mu q_\nu \qquad \text{(B.13)}$$

so that it will automatically give that q^μ on that is zero. So let's say we know there must be a factor like this in front. Then of course we know the dimension of the integral is that much smaller. So the thing to do to remind yourself of that is to take $-p$,

$$N = 4 - \mathcal{N}_g - \tfrac{3}{2}\mathcal{N}_q - p \qquad \text{(B.14)}$$

where p is the known power of the coefficient in front. What I mean by that is the power of external momenta. If you say well I'm going to take the worst case, then p is

zero. We'll consider p zero; I'm going to talk about the cases where it isn't . . . Alright?

Now let's—oh, I had noticed something, that I copied . . . scrap of paper; I'm not going to remember all of it, therefore I can't guarantee it, so you might like to try to prove it. I also got interested in what the order in g is for a given diagram. So I did all my algebra . . . to find the order. And the particular way that I worked out the order was the conventional way, in which the fields that I usually use are replaced by gA so that the action looks like $(\partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu, A_\nu])^2$ *—that's not my conventional way. That's the way I want to calculate the order . . . When I define it that way, and I find the order of g, I find the following rule,*

$$2 \times (loops) + \mathcal{N}_g + \mathcal{N}_q - 2 \tag{B.15}$$

. . . to give you an exercise . . . I don't guarantee it, because I found it on an envelope without checking it. In case you find that useful, maybe you could disprove it or prove it; it would be interesting to try.

Let's find out what kind of diagrams diverge. And to be specific in drawing the diagrams, I'm going to draw the lowest order in each kind. The lowest order in g, not the lowest divergence . . . Then we can represent our diagram by telling how many gluons there are, and how many quarks there are. Here's a little table, and for each case I'm going to draw a diagram to illustrate it. Now if there are no lines coming in, so the diagram has no external lines, well we never have to calculate it . . . So we start with one gluon. And that's a thing that looks like this: the gluon's coming along . . . you can have a loop of gluons, or you can have a loop of quarks; that's a typical diagram. I just draw one typical diagram of this kind.[69] *There would be no electron lines coming in from the outside, and this number* [N] *would be three . . .*

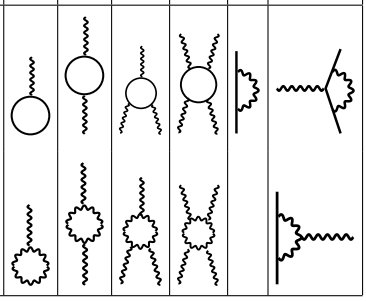
$\mathcal{N}_g$	1	2	3	4	0	1
$\mathcal{N}_q$	0	0	0	0	2	2
N	3	2	1	0	1	0
typical diagrams						

TABLE II: Superficial degrees of divergence, N, for graphs with $\mathcal{N}_q$ external quarks and $\mathcal{N}_g$ external gluons.

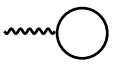
[Question from student: you mean quarks?] *Yes, always I mean quarks, not electrons. I say "electrons" and I say "photons," but I mean gluons when I say "photons," and I mean quarks when I say "electrons."*

Now the next case would be that there were two gluons, and there was something going around. It could be a quark. Or if you prefer, to make it more interesting, make it a gluon, I don't care. Because these are typical, I'm only illustrating. If there are two gluons coming in, as you know, now the divergence is 2. Well you can keep this going. Now we've got three gluons, with something going around here, these are just to illustrate the idea, and there is no quarks coming, the divergence is 1. Or there could be four, [RPF makes sound effects as he draws the legs] *beep, beep, beep, beep,* [and the loop] *loo-loo-loo-loo; now we have four of these, and the divergence is 0. And now if I put five, then it gets convergent. And I stop now; I'm only interested in the divergences.*

So I start now, over again, this time putting in some quark lines. Now you can't just have one quark line because of the conservation of quarks, so the first case you would get would be something like that, it would be no gluons coming in, and two quark lines coming in, and that's [$N =$] *1. And then you could have two quark lines coming in and one gluon line, and you get 0. Now the next thing would be with four quarks, but that's already convergent. And that's the end. Those are the only diagrams—sorry the only types of diagrams—that will bring divergences.*

Mind you, this diagram here, this is lower order, it's plus any internal complications, don't forget. That is, the same divergence will appear to occur if I draw a diagram like this.[70] *It's the same divergence. And that, due to that theorem* [eq. (B.12)] *. . .*

Now we discuss [in more detail] *the individual terms . . . Well if you have a gluon coming in . . . the vacuum here . . .*

 $\sim \langle A_\mu \rangle \,,$ (B.16)

a kind of expectation of the vector potential. The vector potential could be in any direction in the vacuum, and it averages to zero. So this is physically zero, and we never calculate it, because of symmetry. The symmetry is this: we can make $A \to -A$*, the theory is unchanged; not true, not true . . . a little more subtle way; there is a symmetry in here. You change the sign of . . . something like that; anyway there's no direction you can . . . the gluon, so there's no expectation for the mean gluon field. So there's no term, you don't have to worry about it now. In fact we never calculate it . . . there's never any . . .*

Now ordinarily we would expect this

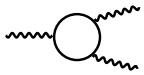
 $\sim \delta_{\mu\nu} q^2 - q_\mu q_\nu$ (B.17)

[69] In my notes I had added the extra examples.

[70] I did not copy the diagram, but the reader can imagine adding loops to the lowest order diagrams.

to be a quadratic divergence, from what it says here [in the table]. *It turns out that the conditions of gauge invariance in the case of quantum electrodynamics for instance, and also in quantum chromodynamics, means that there has to be such a factor* $[q^2 - q_\mu q_\nu]$, *and I'm not going to prove it now, I'm just telling you about this, that gauge invariance makes that* $p = 2$ [in eq. (B.14)]. *And this turns this to a log divergence. In other words we overcount . . . it's not as bad as we think. However if we go to calculate it, if we're not very careful with the cutoff, suppose that the method of cutting off doesn't guarantee gauge invariance, then we can easily get a quadratic divergence . . . we screwed up, okay? But if you do it right, so that you don't lose the gauge invariance in the cutoff process, then you can show that this will only be log.*

Now the divergence of the first power,



(B.18)

really never occurs. Because if we would have–how could we have it? We could have p dot something, like q or something, and then you would have $p \cdot q/(p^2)^2$ *or something. This would be* [in the denominator] *spherically symmetrical, and this would be* [in the numerator] *lopsided. If I change* $p \to -p$ *in the integral I get the same thing with the minus sign. The mean value of* p_μ, *a single p, integrated over all directions, is zero. So for that reason, this "1," here and here both* [in the 3rd column of the table] *the 1 is really equivalent in the end to 0; it's only log divergent. Although it looks like it's linearly divergent, the linear pieces average out, provided that your cutoff isn't lopsided, alright? . . . If you have a reasonable cutoff that's symmetrical, you only get log divergences. And these* [the terms with superficial degree of divergence 0] *are log divergent. And so it turns out that this whole mess, in practice, this whole thing, are all log divergent, if the cutoff has any degree . . .*

[Question from student: does gauge invariance reduce the degree of divergence of some of the diagrams] *There's this* [the vacuum polarization] [student: but not the 3-gluon diagram?] *Yes actually, this cuts this down* [by] *1 . . . The "1" has to be an external momentum. This will produce an effect that is proportional to the original coupling, which is like this. That comes from an AA cross a curl A.*[71] *So there's one gradient that comes on the external line. So it always turns out that this "1," in order to keep the dimensions right, ends up as some external line* [momentum]. *So if we want the momentum of the incoming particle to be in front . . .*

Yes. [Another student: So you do get gauge invariant couplings.] *Something tells us that there will be an extra gradient in front. But it's also true that the mirror symmetry of the . . . the same result. Alright? Another question?* [Shouldn't gauge invariance also reduce the logarithmic divergence in the . . .] *Yes and no. That's . . . more difficult because there's no procedure. This is gauge invariant, but it's coupled directly with A and we don't see any extra gradients. So there's no way to decrease the apparent power. The only way to make it smaller is by having momentum come in front. When you have two quarks—talking about the last diagram—two quarks and a gluon, you're going to imitate a term with a quark, and a quark or an antiquark, and a gluon, and there's no gradient, so it comes out that it's logarithmic.*

Now it is possible to choose a gauge, by the right choice of gauge, you can make any of these damn integrals zero [in terms of the degree of divergence], *at the expense that the others change* [?] . . .[72] *You have to be careful to compute a complete physical process always. You want to make absolutely sure to compute something, the total answer of which is gauge invariant. Then you can't really say—it is possible—you remember all the different propagators we had for the gluon? Well it depends on what propagator you use, whether you use a propagator with* $k \cdot \eta/k^2$ *or whether you use the propagator with* $\delta_{\mu\nu}$, *or still another propagator which is interesting. I wanted to mention this before . . . When we were doing the . . . business, with this gauge for instance; suppose we started with this gauge* $\partial_\mu A_\mu = 0$, *and we come out and we have to do*

$$\int e^{i\int \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu - A_\mu \overset{\times}{} A_\nu)^2}\, \Delta(A)\, \delta[\partial_\mu A_\mu]\, \mathcal{D}A_\mu \quad \text{(B.19)}$$

and I'm not going to write the quark business in. Then we had in addition the statement that there was a determinant $\Delta(A)$, *and this produced ghosts, it was represented by ghosts, and on top of that was a delta function of* $\partial_\mu A_\mu$. *And then we integrate over all A. But I suggested that we get exactly the same* $\Delta(A)$ *here if you try to make this* $\delta[\partial_\mu A_\mu - f]$, *and that the answer was independent of that* $[f]$. *And then suggested further that you multiply by* $e^{-f^2/2}$ *or something* [times] $\mathcal{D}f$. *And the result of that was to bring up a* $e^{-(\partial_\mu A_\mu)^2/2}$*—I'm just outlining what I did. In order to eat, in the square of this* $[(\partial_\mu A_\nu - \partial_\nu A_\mu)^2]$ *the divergence pieces, and then I could show you that the equation of motion which was*

$$\Box A_\nu - \cancel{\partial_\nu \partial_\mu A_\mu} = S_\nu \quad \text{(B.20)}$$

At any rate this term [slashed out] *didn't appear at all, and therefore*

$$A_\nu = \frac{1}{\Box} S_\nu = \frac{1}{k^2} S_\nu \quad \text{(B.21)}$$

So we got the propagator $\delta_{\mu\nu}/k^2$. *Now the interesting thing is what happens if you put a different number here*

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[71] RPF means $\nabla_\times \vec{\mathbb{A}} \cdot \vec{\mathbb{A}} \overset{\times}{} \vec{\mathbb{A}}$; see eq. (8.15).

[72] I don't understand the claim, since the propagator (12.8) (with $\mu^2 \to -k^2$) has the same power-counting properties as usual.

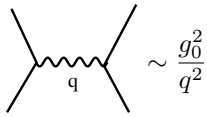
[in the gauge fixing Lagrangian], $a(\partial \cdot A)^2/2$. *And I'll just leave it as an exercise, because if you put a different number in there, you get a propagator of this form,*

$$\frac{1}{k^2}\left(\delta_{\mu\nu} - \eta\frac{k_\mu k_\nu}{k^2}\right) \tag{B.22}$$

where η is not the same as a, I can't remember exactly; η is something like $a/(1 \pm a)$. Ah, it should be when $a = 1$ this disappears; when a goes to infinity, this should go to 1. Because $a = \infty$ brings us all the way back to here; this is Gaussian, such a tightly Gaussian, it's equivalent to a delta function. And it says you calculate everything exactly when $\partial_\mu A_\mu = 0$. And that's a propagator like that $[\delta_{\mu\nu} - k_\mu k_\nu/k^2]$. Now you see that if you take a k_μ of that, you get zero automatically, because the divergence of A_μ is always exactly zero. Well, this general propagator when $\eta = 1$ is called Landau's propagator; when $\eta = 0$ it's called the Feynman propagator. And other η's are possible too; I call it to your attention because it's interesting . . . the effects of this going to zero . . . you've got to be careful what propagator you use . . . what sizes you get for the different . . . It's only when you have a gauge-invariant quantity that you get an answer that does not depend on the . . . propagator you use.

[Some of the lecture was lost during the change of tapes.] *. . . Nowadays it's possible to do all these diagrams and all these calculations on machines, programs for algebra . . . programmed specifically for working on these diagrams and integrals involving quark data, and therefore it gets to be no big deal. You choose a propagator* [gauge], *you turn on the switch, and it does all the 17 diagrams. Whereas by hand, you are happy to discover that by using Landau's gauge, you only have four diagrams; remember I had 17. This would be useful . . . without . . . machines. The reason that four diagrams is better than 17 is mainly, it's impossible to do anything without making mistakes, when you have too many pieces . . . Alright?*

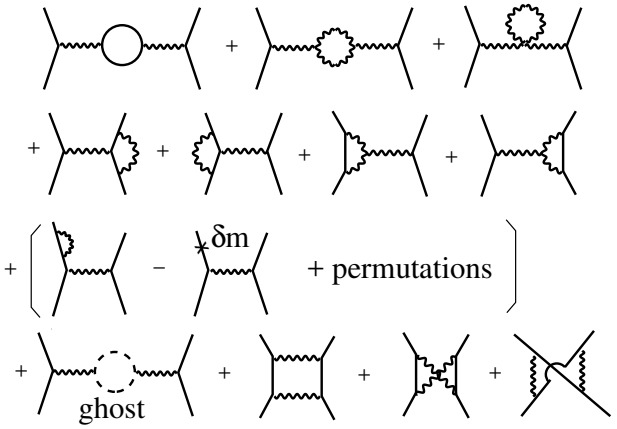
I am now going to calculate, at last; let us talk about the scattering of two quarks. To lowest order we already know that it looks like this



$$\sim \frac{g_0^2}{q^2} \tag{B.23}$$

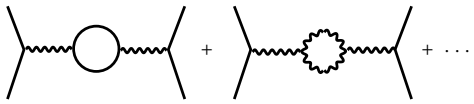
and we computed it. I don't remember what we got for the color . . . but it involves a g^2 and a one over q^2 . . . That's the lowest order. Obviously this is g_{00} . . . now it's going to be α when we're done . . .

Okay, now we get the next order, the g^4. We're trying to get the next order in g . . . So what we do—this is the lowest order—next order, we have a lot of possibilities,

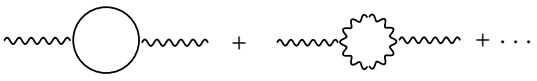


Could be that you had a quark loop here, coming out this way. Could be you had a gluon loop coming out here. Could be that you had a gluon loop like that—what? Yes, with the four gluons that couple there; I believe that's zero when you work it out, but I'm not . . . It's obviously simple. Then there's a diagram that looks like this [vertex correction] *plus one on the other side. Then there's a diagram that looks like this* [3-gluon vertex correction]*—I wish the blackboard went a little bit further, I'll draw it up here, a diagram that looks like this* [quark self-energy correction and mass counterterm] *. . . subtract the effect of this, this thing is divergent, the correction to the quark mass, you just have to subtract it. Alright now I've drawn all the diagrams except the mirror image of this one, the mirror image of this one, this* [self-energy] *could be here or here or here . . . Alright? Now any other diagrams you can think of?* [Student mentions the ghost.] *Ah yes, I'm sorry, the ghost. Important, important, the ghost, the ghost. Very vital. Ghost, ghost, ghost, ghost. Thank you, yes. Anyone else think of some oversight? . . .*[73]

Alright. Now the situation is, a lot of people will like to compute this



and say, well this is just a gluon going along, so don't do all this, just do this,

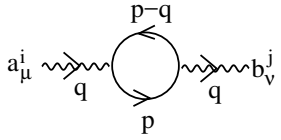


on the gluon . . . [Student (me): there's a box diagram with . . .] *Just a moment. Just let me finish this part. Just do these* [now including the gluon loop contribution to the vacuum polarization]. *You can't, because this depends on what kind of gauge propagators you use, it's not*

[73] I realize that the box diagram is missing a few minutes later in the lecture and interrupt RPF.

a gauge-invariant process . . . You have to finish it by having quarks make the gluons . . . that's why I made it more complicated. The answer is that depending on what kind of propagator you use, you get different answers for that. It's only when you put the whole mess together that you get an answer that's independent of what propagator you use. Somebody was going to say something. [Student: there's a box diagram too, where you've got two gluons connecting the quark propagators.] *Of course, sure there is. Yes. And crossed, right?*[74] *I thought we'd find something . . .*

What we can do is do one as simple as possible, and then say the rest of them, you know the rules for putting this in and the rules for putting that in, they follow the same kind of . . . the labor is enormous . . . how to sum over the colors, various things . . . If you had to do it, you would do it. Or find a book that will do it, okay? But in order to understand the nature of the results that we're going to get, I'm going to take only one case. I'm going to take the dullest and the simplest case, this one,



. . . The others teach you a little bit, but . . . you've probably done this . . . So here we go. Alright? . . . I'm going to have something going around here; let's say that the momentum coming in here is q, and this has a momentum p let's say, and then since the momentum coming in is the same as the momentum going out . . . If you don't know what I'm doing, then it's because you didn't take a course that's supposed to be a prerequisite for this, having to do with perturbation field theory, and you're going to have to learn it. Alright? If you have taken such a course, this will be very boring, and I'm sorry, but I'll go as fast as I can, and hope that you . . . will stop me and ask some questions . . . Alright, now here we would have a quark going around, and we would have a coupling in here, which is a γ_μ in the direction of the polarization of this, so let's say that a is a vector polarization, and let's say a^i is the color of the gluon. So this would be γ_μ and it would be multiplied by a^i_μ. But there would also be a λ_i matrix for the color matrix that this couples with,

$$g_{00}^2\left(-\mathrm{Tr}\int\frac{d^4p}{(2\pi)^4}\,b^j_\nu\frac{1}{\not p-\not q-m}\,\frac{\lambda_j}{2}\gamma_\nu\frac{1}{\not p-m}\,\frac{\lambda_i}{2}\gamma_\mu a^i_\mu\right) \tag{B.24}$$

Now the next thing happens is that this damn thing propagates, the quark propagates around to here, and the fact that they're propagating, one over $\not p$ minus the mass of this quark. Sir: [student: Do you mean to have $\lambda_i/2$?] *Yes sir, I do, I do. Yes, thank you. Then comes this baby which I'll call b, and we can say the polarization is γ_ν, we're going to be multiplying by b_ν and by j. And then we'll also have a λ_j, for the color business there, and then we'll have—I didn't leave enough room did I?—well maybe I just about did, I have the propagator p dagger minus q minus m—oh, I forgot the 2 again, I shouldn't do that. And I have two g's for the couplings at either end. And those are the kind of g's that I was tagging g_{00} . . .*

Of course, that looks like this is second order, but I didn't put the rest of these two lines on [the external quarks], *and when I did that there would be two more g's, and this is important. The first diagram which got two g's and the second diagram has four g's . . . Now, I have to sum over everything. I have to sum over all the possibilities for the momentum of the quark loop, and I have to sum over the colors, and the spinors. Well, first the spin. When you come around and come back, you've got all these matrices, and it becomes a trace. So really it should have been a matrix trace, a trace for the gamma matrices,*

$$\underbrace{\mathrm{tr}\left[\frac{1}{\not p-\not q-m}\gamma_\nu\frac{1}{\not p-m}\gamma_\mu\right]}_{\mathrm{tr}\left[\frac{(\not p-\not q+m)\gamma_\nu(\not p+m)\gamma_\mu}{[(p-q)^2-m^2][p^2-m^2]}\right]}\qquad\underbrace{\mathrm{tr}\left(\frac{\lambda_i\lambda_j}{4}\right)}_{\frac{\delta_{ij}}{2}} \tag{B.25}$$

$$=\Big[(p-q)_\mu p_\nu+(p-q)_\nu p_\mu\;-\;\delta_{\mu\nu}(p\cdot(p-q)-m^2)\Big]/\ldots$$

. . . In addition, we'll have a matrix operator for the color, and then it carries all the way around here and goes to this color, and then it carries all the way around so we get another kind of a trace, this time on the colors. So this is a trace on the gamma matrices, you might say, and then there's another . . . trace on the color indices, of $\lambda_j\lambda_i$ over 4. And this one is easy, we know the color of this—without the 1/4—is $2\delta_{ij}$, so that's $\frac{1}{2}\delta_{ij}$. So the first thing it tells us of course is that the color of the quark that comes out must be the same as the one that went in. That's the conservation of color; if it's a red-antiblue gluon here, it will be a red-antiblue gluon there . . .

This particular trace [the Dirac trace], *the famous way of handling that, is to multiply numerator and denominator by $\not p-\not q+m$ and by p dagger plus m, and then in the denominator, you'll have the rationalized thing there, $(p-q)^2-m^2$ and p^2-m^2, and then there will be this factor we had before, δ_{ij} over 2. Alright? . . . Any question up to here? In fact it's the same as the corresponding correction in electrodynamics, except there's some slightly different number . . . from the colors. This trace here that I've written can be directly simplified; this trace is exactly the same as this dot this plus the other way around . . . minus . . . it's all memory . . .*

So the net result is that effectively I have to do this integral, an integral that you must have learned about when you were doing electricity, QED, so I turn over here to do the integral. But before I do the integral, I look at it, and

[74] There are many other crossed diagrams missing.

notice how it diverges. We can see by counting there's 2 p's in the numerator, there's 4 p's in the denominator, there's 4 p's in d^4p. That cancels but you've still got 2 so it's quadratically divergent. In exact agreement with the prediction. But, now the trickery, the method for doing this like this, is to have a whole list of integrals . . . a list like this—I'll do the integrals in a minute—but things like this,

$$\int \frac{d^4p}{(2\pi)^4}\frac{1}{(p^2-m^2)^3} = \frac{1}{16\pi^2 i m^2} \tag{B.26}$$

or something; I'm not chasing that, alright? This permits you to do anything that has powers in the denominator, by integrating over . . . But this [(B.25)] *has two different kinds of powers in the denominator, so there is an invention for putting that together, which runs like this:*

$$\int_0^1 \frac{dx}{[ax+b(1-x)]^2} = \frac{1}{ab} \tag{B.27}$$

The integral from 0 to 1 dx . . . and this you can verify directly. And therefore if you take the product of two pieces, you can write it as one denominator.[75] *I copied it from Schwinger, actually, I cheated. He had another way of doing it which was extremely clever. Involving a Gaussian integral. I noticed that I could eliminate one step . . . and make it look . . .* [laughter].[76]

So therefore . . .

$$\frac{1}{(p^2-2p\cdot q+q^2-m^2)(p^2-m^2)} = \int_0^1 dx\,\frac{1}{[p^2-2p\cdot qx+q^2x-m^2]^2} \tag{B.28}$$

Alright? Alright, . . . what we're going to replace that thing by, and so we would have this integral from 0 to 1 dx, to be done, later, and then we'll have the integral d^4p, and then we have something like . . .

$$\int_0^1 dx \int \frac{d^4p}{(p^2-2p\cdot qx+q^2x-m^2)^2} \times \left(2p_\mu p_\nu - p_\mu q_\nu - p_\nu q_\mu - \delta_{\mu\nu}(p^2-p\cdot q-m^2)\right) \tag{B.29}$$

Alright. Now we have to do this [p] integral. But we can't, because it's divergent. The first thing that's a good idea to do, always, is to shift the p, let me shift p, let $p \to p+qx$. Then we have

$$\int_0^1 dx \int \frac{d^4p}{(p^2+q^2x(1-x)-m^2)^2} \times \Big(2p_\mu p_\nu + (p\cdot q\ terms) + 2q_\mu q_\nu\, x(1-x) - \delta_{\mu\nu}(p^2-q^2x(1-x)-m^2+p\cdot q\ terms\ )\Big) \tag{B.30}$$

I just completed the square here. And then there are some terms with $p\cdot q$, I don't know what they are . . . they are linear in p, you'll see why I don't care about them in a minute . . . Anything linear in p that I didn't bother to write out, the reason is that when I integrate over all directions of p, the plus and minuses are going to go out, and since I'm going to get nothing from them, I didn't bother to be careful. Alright?

Now I have cheated. Why? We made a mistake. Because, the integral is divergent; I didn't tell you how to make it convergent. Therefore this business of shifting p by qx—how do I know that the integral—that you can shift p by qx and get the same result? I must insure that my method of renormalization, whatever the hell I'm going to use, has that property that if you shifted the p, it would be alright, okay? So that's one thing. The second is, what about the integral over p at zero? That's right, if the method of renormalization is round, okay, but round against what? Against the shifted p? If you say it's round against this p, it's not round against the shifted p. You know what I mean, symmetric in both directions. It's cheating. So only if I made a method of renormalization, I mean a method of cutting off the integral, specific, and carefully, can I really do these things which I've been doing.

Now it happens to turn out that one method that's been invented for this kind of loop of quarks or electrons is this: you subtract the same expression with a larger value for the mass of the electron. In other words, the method of renormalization, I should have at least specified before I made those steps. And the method that I want to use is going to be this one. You take the value and then take with m^2 replaced by a bigger m^2, which I like to write as $m^2 \to \Lambda^2 + m^2$, and subtract. That's the scheme. That scheme, it so happens, will permit the steps which I did of shifting. It was all right. But it's very easy to slide off the wagon and make operations which are not quite right until you specify the right way . . . You see, this method of subtraction maintains the gauge invariance for this diagram. Because if this were electricity, everything is exactly the same except for a number, and if this is electricity, if this is any mass whatever, this is a gauge invariant integral, and the gauge invariance is maintained by subtracting the same expression. To show you that it isn't necessarily obvious what to do, an early worker in the field first proposed or tried to subtract from this propagator the propagator with a different mass. In other words, to use for the propagator

$$\frac{1}{p^2-m^2} \to \frac{1}{p^2-m^2} - \frac{1}{p^2-(\Lambda^2+m^2)} \tag{B.31}$$

That doesn't work. That doesn't keep the gauge invariance. Pauli and Villars pointed out that if you subtracted the whole thing, the whole closed . . . [amplitude] *then you maintain the gauge invariance. So you see it's easy to . . . if you don't maintain the gauge invariance . . . electrodynamics . . . Pauli-Villars . . . and that's the way if you do this then all these things are legitimate.*

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[75] Here RPF makes a joke that elicits laughter, something like "Classic, right?"

[76] Perhap RPF is saying something like "and make it look like my own idea."

Okay?

Now I'll show you a very interesting—is that okay, you had a question? It's nerve-wracking, alright. But it's been straightened out, in the case of an electron. I want to point out, that when we come to the second diagram here, with the gluon, it's more subtle and more complicated. And the exact way to do it is very hard, and it took a lot of finagling around to get it right, when I first tried it. But I was able to guess and push and hammer. In the meantime, another method of cutoff was invented, which is called the dimensional renormalization, invented by Wilson and 't Hooft . . . for which gauge invariance and covariance in space in four dimensions are automatically maintained in [d dimensions]*, not a chance of losing it, so that it's a good scheme. We don't have to have the old-fashioned hammer tricks. Here we know a good trick that will work, and I wanted just to point out . . . Now I would like to point out—I think we could almost see—Now I want to show you something. Suppose we had a method of renormalization, that we knew was going to be symmetric, and everything is going to be alright. Then, I claim this integral,*

$$\int d^4p \, \frac{p_\mu}{p^2 - m^2} = 0 \tag{B.32}$$

if we could renormalize, to make it finite anyway, would have to be zero because of the asymmetry. Provided we had a good method to protect the asymmetry, and subtracting this thing with a different mass obviously does that. That's it, no problem. Now differentiate both sides of this . . . Now what I want to prove, let's see . . . If you differentiate inside the integral with respect to [p sub]ν*, you'll get*

$$\int d^4p \, \frac{\partial}{\partial p_\nu} \frac{p_\mu}{p^2 - m^2} = \int d^4p \, \frac{\delta_{\mu\nu}(p^2 - m^2) - 2p_\mu p_\nu}{(p^2 - m^2)^2} = 0 \tag{B.33}$$

It's not hard to prove, by the same kind of symmetry, one way or another, you might not like the way I did it, that this is also zero.

[Student: what did you differentiate?[77]] *I differentiated the integrand with respect to p_ν. Isn't it legitimate to differentiate the integrand if you're going to integrate it back anyway? Well, if you don't like it this way, then another way to do it is to shift p, and differentiate with respect to the shift in p. You start out with some kind of thing like*

$$\int d^4p \, \frac{p_\mu}{(p-a)^2 - m^2} \;=\; a_\mu \int \frac{d^4p}{(p-a)^2 - m^2}$$

which is just this thing [(B.32)] *with p substituted with $p - a$. And then differentiate both sides of this expression with respect to a_ν; then put $a_\mu = 0$. Alright?*

[77] I reconstructed the previous equation from the tape rather than my notes; apparently it was less explicit, leading to this question.

I write this particular thing by putting it all under the same denominator, and putting this here [arranging the numerator so that the first term is like in (B.33)]*:*

$$\int \frac{d^4p}{(p^2+q^2x(1-x)-m^2)^2} \Big[2p_\mu p_\nu - \delta_{\mu\nu}(p^2+q^2x(1-x) - m^2) - 2q_\mu q_\nu x(1-x) + 2\delta_{\mu\nu}q^2x(1-x)\Big]$$

So we have this sort of general statement that it is a displaceable method of doing things. You can think of it as a shift of the origin; the method of renormalizing doesn't have anything to do with it. The question is, if this were automatically zero. And then if we look at this we expect that . . . we have just the right combination: we have $2p_\mu p_\nu$ and $\delta_{\mu\nu}$ times the denominator—watch out on that sign! [The q^2 term in the numerator has the opposite sign to that in the denominator.] *This part's okay but this is wrong. Well I'll fix it; I'll put plus and I'll make it minus 2* [in the second line]*. Alright, that's not wrong. Now this thing, times the delta, plus this one, go to zero . . . by the argument about the way to do it . . . in the integral . . .*

The result of that is that all of this can be . . . the net result of the whole thing, altogether, is

$$2(q_\mu q_\nu - \delta_{\mu\nu}q^2) \int_0^1 x(1-x)dx \int \frac{d^4p}{(p^2 + q^2x(1-x) - m^2)^2} \tag{B.34}$$

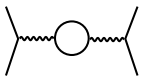
and I'm close to the end of the hour . . . I wanted to subtract—this is only a logarithmically divergent thing—and I subtract the upper limit and so on. It's going to introduce something that goes like the logarithm of this cutoff divided by some pole mass which is practically the q^2 of the quark. It's a little more complicated than that and I'll finish it next time. What I'm trying to say is that we can get to this thing in front, and that is necessary for gauge invariance, because q dot [the prefactor] *is zero . . . The current is conserved.*

Now let me, since it's just a few more minutes to ten, just to remind you . . . I'll explain that this means that the vector potential that is coming in and out . . . finish the job next time . . . and also next time I'll start to explain . . . dimensional renormalization, which is so handy, because the old-fashioned way you had to do a lot of thinking, trickery, to make sure you didn't screw up . . . invariance.

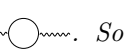
Appendix C: Transcription: Renormalization, continued (1-14-88)

We were doing one of a number of diagrams that have to do with the correction to scattering of two quarks. The scattering of these two quarks behaves as—g is the coupling constant—something like g^2 over q^2 in the first order. Or that's sometimes called second order because it's in g^2 . . . First order in g^2. That's straightforward. We

want to get corrections for it. And the corrections appeared to be a large number of diagrams which I wrote ... and you'll see that there's a g here and here and here and here,



$$= J_\mu \, g_0^4 \, \frac{1}{q^2} \, B \, \frac{1}{q^2} \, J_\mu \qquad \text{(C.1)}$$

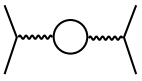
where $B =$  *. So this is going to be a correction which will be of the order* g_0^4*. There will be two propagators* $1/q^2$ *here and* $1/q^2$ *there, and so on. So it will be* $1/q^4$ *multiplied by an integral,*

$$B = (\delta_{\mu\nu} q^2 - q_\mu q_\nu) I, \qquad \text{(C.2)}$$

$$I = \int_0^1 dx \, x(1-x) \int \frac{d^4p/(2\pi)^4}{[p^2 - m^2 + q^2 x(1-x)]^2}$$

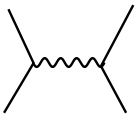
What we're going to discover is that this is—sorry, not by the integral, but by B*, the bubble ... I left out all the indices, the colors and all that. You can go away and discover that the colors are all ... multiplied by a certain integral ... in this integral there is a number of constants which I inadvertently, carelessly dropped—twos and* π*'s and things which presumably you can calculate ...* [78]

Now, the currents $[J_\mu]$*, which I didn't write here, which are operating here and here ... this bubble, when acting on the current, the* q*—the current is conserved—and the* q *acting on the current is zero, so that* $q \cdot J = 0$*. The bubble gives you* $\delta_{\mu\nu}$*, which means that these two currents are in the same direction, and this* q^2 *in the bubble eats one of these* q^2*'s. So this thing turns into*



$$= J_\mu \, g_0^4 \, \frac{1}{q^2} \, I \, J_\mu \qquad \text{(C.3)}$$

So therefore, it's the same form as this [the tree level contribution]



$$= J_\mu \frac{g_0^2}{q^2} J_\mu \qquad \text{(C.4)}$$

except we have g_0^4 *times the integral up here instead of* g_0^2 *in the correction. So the easy way to think about it is it's a correction to this coefficient at the top,*

$$g_0^2 \to g_0^2 + g_0^4 I \qquad \text{(C.5)}$$

Alright? Are there any questions about that?

In getting to this form we did a little hocus-pocus about correcting, shifting origins and this and that, and talking about tricks to get rid of the quadratic divergence which originally arose. However we still have a divergence and we have to talk about how to handle it. And the particular rule that works for log divergences in quark loops is to subtract—the rule I'm going to use here, now— is the method of subtracting the same result for different masses. But I will describe, perhaps today, but later, the method called dimensional renormalization, which is to change this d^4p *to* $d^{3.9}p$*. This is a function of only* p^2*, so* d^4p *is something like* $p^3 \, dp$*, with a coefficient that is* π^2 *or something, that depends on the number of dimensions; in three dimensions it's* $p^2 \, dp$ *times some other number,* 4π*. And in* d *dimensions* d^dp *is* $p^{d-1} \, dp$ *times some coefficient involving Gamma functions of* d *and so on ... That's all there is to it—mainly that's all there is to dimensional renormalization; to use* $d = 3.9$*, and then* 3.99*, go to the limit. Alright? That's what it's about. However it's very pretty and I must have spent a lot of time because I enjoy it ...*

Anyhow, the way of renormalizing is to subtract—this is what we're going to do now

$$\frac{1}{(p^2 - (m^2 + \Lambda^2) + q^2 x(1-x))^2} \qquad \text{(C.6)}$$

[from the integrand of I]. *That will make the results convergent ... It would be—another way to make this subtraction is to consider this integral as a function of* m^2*,*

$$I(m^2) = \int_0^1 dx \, x(1-x) \int \frac{d^4p/(2\pi)^4}{[p^2 - m^2 + q^2 x(1-x)]^2} \qquad \text{(C.7)}$$

and then consider taking $I'(m^2)$*, the derivative with respect to* m^2*, and integrating that*

$$-\int_{m^2}^{m^2+\Lambda^2} I'(M^2) dM^2 = I(m^2) - I(m^2 + \Lambda^2) \qquad \text{(C.8)}$$

Let's take the derivative with respect to m^2*, but we're going to have to have a variable for it, so let's call it* M*; integrate that with respect to this* M^2 *... you can certainly do this ... This is a trick that I wanted to use ... converges ... Well of course I need to differentiate and then put it under the integral sign, calculate the integral and then ...*

... This particular method would produce this final result without all those tricks about shifting and so on; all those could be perfectly done by ... and not notice all that stuff ... all that stuff that I did for quadratic divergences, it's also taken care of ... I did that to show you how in cases where there's some confusion, it is always possible to get an answer. People were very clever to squeeze answers out of these ...

The logarithmic divergences are always much easier to handle, much less uncertain than quadratic and higher divergences. And when we were doing this stuff for example we found that one of these things would produce a logarithmic divergence directly. So another thing to do is to compute a process that we know produces a logarithmic divergence, and then have no more trouble, and use

[78] I had added the $1/(2\pi)^4$ factor as an afterthought in my notes.

gauge invariance to get the terms in front. But with dimensional renormalization, you don't need any guessing . . .

Okay, we're going to do it this way [by Pauli-Villars subtraction] *here. So first I'll differentiate that* [(C.7)] *and I will find myself needing to do the integral*

$$\int_{m^2}^{m^2+\Lambda^2} dM^2 \int_0^1 dx\, x(1-x)\;\; 2\;\; \underbrace{\int \frac{d^4p/(2\pi)^4}{[p^2+q^2x(1-x)-M^2]^3}}$$

$$\int \frac{d^4p}{(2\pi)^4}\frac{1}{(p^2-L)^3} = \frac{1}{32\pi^2 iL} \tag{C.9}$$

And now at the end when I'm all done I have to integrate with respect to M^2 . . . Alright, so that's where I'm at, you see where I got that: differentiate that with respect to M^2, the second power . . . now it's got the third power. Now of course this integral will present no divergence because there are six p's in the denominator and four in the numerator; then this integral can be done. We can do it in lots of ways, and I'm not going to bother . . . Anybody who's ever done anything in perturbation theory is always going to put this very same integral, sooner or later. It's the integral of $d^4p/(2\pi)^4$ divided by p^2 minus something, cubed, and it's equal to [one over] $32\pi^2 iL$ *or something like that. Factors of 2 or so I'm not going to . . . I can't remember and I didn't bother to look it up . . .*

This then, this integral here then—I'm keeping these lines this way so that you don't have to write that over and over and over—this piece becomes

$$\int_{m^2}^{m^2+\Lambda^2} dM^2 \;\; \frac{1}{16\pi^2 i}\,\frac{1}{M^2-q^2x(1-x)} \tag{C.10}$$

$$= \frac{1}{16\pi^2 i}\ln\left(\frac{m^2+\Lambda^2-q^2x(1-x)}{m^2-q^2x(1-x)}\right)$$

Okay, now the idea is that Λ is the cutoff, it's supposed to be higher than any of this part with the mass, even q^2. We only want this theory as $\Lambda\to\infty$. As $\Lambda\to\infty$ then, relatively speaking we can drop this $[m^2-q^2x(1-x)]$. Of course we get an infinite answer because we have a divergence and that's where all the trouble began. What happens to that infinity? It's fixed by changing the coupling constant g_0 with Λ, by making g_0 a function of Λ, so that the variation of the first order takes away the . . . if you use different Λ's you use different g_0's at the end.

I'm going to continue this calculation disregarding the masses. Suppose you have a large momentum transfer, and disregard the mass. The purpose of this is only to do the arithmetic; if you want to you can always do it with [nonvanishing mass]*. . . I'm going to disregard this just so to take a simple example where q^2 is much larger than the m^2 . . . It looks like it's dangerous because when x is small, even if q^2 is large, maybe that* [the mass] *is important, but it turns out in the log it don't make any difference, but anyway you do it with m^2, I don't want to do it with m^2. I'm only illustrating, explaining what comes up . . . So I'm going to write this as*

$$\int \frac{dx\,x(1-x)}{16\pi^2 i}\left[\ln\left(\frac{\Lambda^2}{-q^2}\right)-\ln x-\ln(1-x)\right]$$

$$= \left(\frac{1}{6}\ln\left(\frac{\Lambda^2}{-q^2}\right)-\frac{5}{18}\right) \tag{C.11}$$

We suppose that $-q^2$ is positive, actually. It's going to be a momentum transfer for the scattering . . .

Now I have to integrate this and I'm almost finished, see? So I have to integrate $x(1-x)$. That's a constant [meaning that $\ln(\Lambda^2/q^2)$ does not depend on x] *. . . that's well within my power. $x-x^2$* [integrates to] *$x^2/2-x^3/3$ that would be the integrand, and if I put x from 0 to 1—no the integrand is the differential of that—I get* $1/2$ *minus* $1/3$ *is* $1/6$*. It's going to come out to* $1/6$*. Silly now, because I've lost the constant . . . so I have the logarithm, I'm just trying to keep things that are relevant . . . Now I have to do the log of x times this, which I'll do by parts. And I get* $1/4$ *minus* $1/9$ *actually, when I do it by parts I get some kind of number here, something like* $5/18$*, okay? Coming from integrating those logs, which are easy to do, and I'm sure you'll . . . Yeah,* $5/18$*. Anyway you get some constant. And that's it.*

And that's the integral "I," and it will have the Λ there. How are we going to look at the physics? Assuming the physics is right, and we're going to get an answer, now we put that back: this "I" goes back in here [eq. (C.3)]*, right? So, I'm just going to write the coefficient of the term . . .*

$$J\frac{1}{Q^2}J\left[g_0^2 \;+\; \frac{g_0^4}{16\pi^2}\left(\beta_0\left(\ln\frac{\Lambda^2}{\underbrace{-q^2}_{Q^2}}\right)+a\right)\right] \tag{C.12}$$

$$\text{where } \beta_0 = -\frac{2}{3}n_f \qquad \text{and } a=\frac{10}{9}\,n_f \tag{C.13}$$

g_0^2 plus g_0^4 times the various numbers of π's, which I have recovered by looking at the answer in the book, alright? Times a certain constant [β_0] *times the logarithm of Λ^2 over $-q^2$ plus another constant, where for us, β_0 it turns out is $-2/3$ when we put it in this form. Alright? And the "a" for us is equal to—this is* $5/3$ *of that,* $5/3$ *minus* $2/3$*, probably something like plus* $10/9$*—highly questionable . . .*

Now first of all, to remind us all that $-q^2$ is positive, let's call it Q^2. So I'm going to make it like that. This [eq. (C.12)] *is all multiplied by $1/Q^2$, and by the currents and so forth in the final interaction. That's what comes out, alright? . . . The reason I wrote it this way is that there's going to be more contributions that come from the other diagrams that we haven't worked out. And I'm going to have to add those in when I discuss . . .*

Oh—[the factor of n_f was not written in eq. (C.13) at first] *there's more than one flavor of quark, and each flavor of quark makes a loop, and each one those is the*

same as this one. And insofar as Q^2 is large enough to neglect the mass of the quark, insofar and therefore for the first few, certainly for the u and the d and very likely for the s, maybe for the c quark, there will be a certain number of flavors that we would use, that contribute to this formula; it would be the number of flavors whose mass is less than Q^2. The flavors with masses higher than Q^2 are not much contribution. The flavors that are in between, then you just have to do the integral better and so on. So put the number of flavors [in (C.13)]. *Alright?*

It looks like this answer means that the probability of scattering will depend upon the cutoff Λ—there it is, explicitly there. And therefore our original program, which was that we were going to calculate somehow in quantum chromodynamics and make predictions, but we found our theory was divergent, and what are we going to do? Well we have a . . . But the trick is to arrange that it doesn't depend on the cutoff, by supposing that g_0^2 is chosen, for each Λ that you choose, you must take a different g_0. You must choose a g_0 which is a function of Λ, chosen so that the answer to a physical question does not depend on Λ. We need a formula for that to do that. Well this is very hard to figure out here by looking at this thing, what kind of a shenanigans, how you're going to vary g_0 to get rid of this ln. *Of course this could also be written as g_0^2, and to the same order as I have it here, it's convenient to write it this way,*

$$\frac{g_0^2}{1-\frac{g_0^2}{16\pi^2}\beta_0 \ln\frac{\Lambda}{Q^2}} = \frac{16\pi^2}{\left(\frac{16\pi^2}{g_0^2}\right) - \beta_0 \ln\frac{\Lambda^2}{Q^2}} \qquad \text{(C.14)}$$

There's one other term I forgot about, this $(16\pi^2)$*, this is very much like . . . That's equivalent to this approximation* [Taylor expanding to first nontrivial order]*; we're only worried about this log part now; the "a" is something else we have to work out a little more accurately. To this order, as far as the* $\ln\Lambda^2$*, it's like this . . . which could also be written* $1/g_0^2$*, or I would prefer to put my* $16\pi^2$ *here, so perhaps there's a* $16\pi^2$ *here, minus* $\beta_0 \ln\Lambda^2/Q^2$*. Now this* $[g_0^2]$*, we can suppose depends upon* Λ*, and we have to make this* [the right-hand side of (C.14)] *not depend on* Λ*. If we make* $16g_0^2$*, if we arrange all the time that* $16\pi^2$ *over* g_0^2*, which is by the way* $g_0^2/16\pi^2$ *. . . old notation where I had forgotten the* 4π *squared, so you can remember this in connection with the previous lecture, without seeing that lousy* 4π *squared all the time. This, if we suppose this, which is defined as this,*

$$\frac{16\pi^2}{g_0^2(\Lambda)} \equiv \frac{1}{\hat{g}_0^2(\Lambda)} = \beta_0 \ln\frac{\Lambda^2}{\lambda_P^2} + \text{const.} \qquad \text{(C.15)}$$

is equal to, when we vary the Λ*, we make sure that this is some kind of constant, well we should make this is* $\ln\Lambda^2$ *plus a constant. And since it's a constant, we can put anything down here, that we want . . . This will cancel the* $\ln\Lambda^2$*, yes? No, you've got to have a* β_0 *in here* [initially forgotten in the above formula].

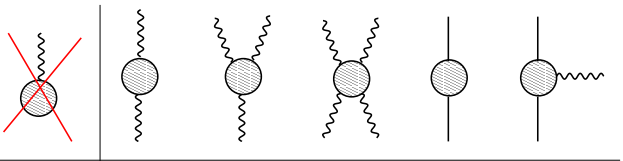
So if we suppose our g's are chosen like this, then we'll get an answer that is independent of $\ln\Lambda^2$*; that's the trick. When we do different degrees of convergence with the cutoff, and we change the cutoff, and we change g^2 that we use appropriately for that cutoff, then we can arrange the whole thing so that it doesn't make any difference where that cutoff is, that's the miracle of this theory, and that's why we have a theory. Because otherwise you would have predictions that would depend on still another parameter which is the cutoff, where we have to write our theory with explicit formulas for the cutoff . . . This way, we don't; we just have to say that it's going to happen, and hope that it does, it's been proved that it does . . . Alright?*

Well this is explaining the machinery that we're going to . . . The thing is we haven't gone to the next order, g_0^6, and discovered that this isn't quite enough; that there has to be a log-log term there. I'll explain why there's a log-log term . . .

Yes? [Question about the constant in conjunction with λ_P.] *I don't know that constant. It's arbitrary, you choose anything you want. Later on we chose, specifically, to make Politzer's λ, to make that constant serve for a special method of cutting off dimensional renormalization, which is not what I'm going to use, so the π, the* 10/9 *and all that stuff is changed, because of the kind of handling of the integral. And because it's a different—yes, if you had decided that the method of cutting off was going to be the Λ method, then in that Λ, the way I did it, changing the masses of the quarks, that would be enough . . . then a more convenient λ Politzer might be to make that zero* [referring to the constant in (C.15)]. *. . . Anything would be alright. Somebody has to make a choice somewhere. . . . There's a choice to make that constant zero for dimensional renormalization . . . Let's take the case of the lattice. You say that the Λ corresponds to the wavelength of the spacing. So is Λ* [equal to] *$1/a$ or $2\pi/a$? One guy does one way, another one another way; all he does is change the scale of Λ. So you need to put that as the log of 4π squared over here, or you can change the definition of this . . . That's why I like to put the constant here at the end . . . Any other questions?*

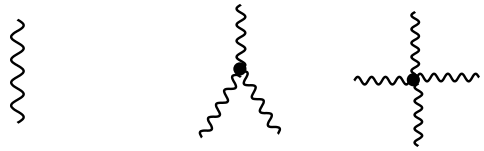
[Question: why does this procedure work for all processes, using the same $g_0^2(\Lambda)$ for all of them?] *That has been proved. That we had to assume, that the theory was . . . It's not obvious at all. It turns out that . . . it's not true only of the calculation at second order, but at the next order also it's independent of the process. Only beyond that does it become dependent on the process. And that's connected to the discovery that the rate of change of the $1/g_0^2$ with respect to Λ computed as a series . . . Oh I shouldn't say it that way, I should have used a real process. The first two terms . . . how we know is something I didn't prove.* [Student interjects, I think it is me: Isn't the answer to that question the fact that you have a finite number of . . . with primitive divergences?] *Yes, we'll discuss that. Thank you. Yes, thank you. That's good, let's discuss it. He's got the right answer . . . We worked*

out the various divergences.



N (Degree of divergence)						
naive	2	1	0	1	0
actual	0	0	0	0	0

This one [the tadpole] *vanishes. Then there was divergences of this kind . . . two things coming out, never mind what's in here; there would be three gluons, remember this little table I made? With the degree of divergence . . . There was a table that said how many glues there were and how many quarks there were on the outside lines, from which we calculated as 4 minus this* [number of gluon lines] *times* [minus] 3/2 *that* [the number of quark lines]. *And the result for N here was 2, 1, 0, 1, 0, and everything else convergent. Now using gauge invariance you can show always that in this case* [$N = 2$] *there have to be two q's outside, in front . . . and therefore this is really . . . 0, logarithmic divergence. And this one* [the linear divergence] *by symmetry again, a single power of momentum . . . this is also a logarithmic divergence, so is this. So altogether they're all logarithmic divergences, it's no big deal. So these are the kinds of things that are divergent. This* [the vacuum polarization] *will always have* $q_\mu q_\nu - q^2\delta_{\mu\nu}$, *something like that. Because of dimensions however, you see there must be, the whole thing has dimension one so there has to be at least one q sticking out in front . . . these things* [the two-point function], *insofar as they diverge, at high momentum, they must be numbers times these two A's with q's in it. And if you call the vector potential of this A, and* [of that] *A, and you make the Fourier transform back again, in other words, if we hadn't had those cutoffs and I've just got quarks in here—er gluons in here—then this bubble . . . for having a* q^2 *in front, which I illustrated here . . . double gradient on the A. And because of gauge invariance, the result must be*



$(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \qquad A_\nu^\times A_\mu^\cdot \partial_\mu A_\nu \qquad A_\nu^\times A_\nu^\cdot A_\mu^\times A_\nu$

Likewise this term [the three-point function], *involving three potentials, is equivalent to the effect of some direct contact—the divergent piece—is equivalent to some kind of contact which involves three A's and one gradient. But because of gauge invariance, the only thing you can write that has that property is this kind of thing, to go along with this one* [the kinetic term]. *And furthermore, these four, has four A's, and they will turn out to be of this form. Not only that, but the* [coefficients] *of every one of these things will be adjusted just right so that the combination of these things with their coefficients, all the coefficients will be right, so that this is equal to*

$$\ln \Lambda^2 \, F_{\mu\nu} F_{\mu\nu} \tag{C.16}$$

times a number, which involves this divergent log . . . The log divergent part looks like this. So if I had computed this one or this one, I would have gotten the same result. The reason it has to have this form is gauge invariance. If I did not destroy the gauge invariance by the cutoff method. Now the particular cutoff method I used was forced to not spoil the gauge invariance . . .

Likewise this thing [the quark self-energy] *is going to involve two* ψ*'s. And this one* [the vertex correction] *is going to involve two* ψ*'s and an A. This one* [the vertex correction] *corresponds to changing the coupling constant here, and this one* [the self-energy] *corresponds to this—there is a term* $\bar{\psi} m \psi$ *. . . corresponds to the idea of changing the mass of the quark. Well I have just erased something here that I think I need: when I change* Λ*, I'll change g because I'm going to suck that number into the original zero order* $(1/g_0^2) F_{\mu\nu} F_{\mu\nu}$ *which is what I started with, and this gives corrections—are going to produce corrections—to this thing times* $F_{\mu\nu} F_{\mu\nu}$*, and I'm going to say "Oh. I could have started with a g, I'll make this g change to eat that number." In other words* $1/g^2$ *will vary in such a way to eat that number. And that's what those formulas* [(C.15)] *are for* $1/g^2$*. They're just designed to eat these logs. The gauge invariance enables you to know that all of these are going to all go together . . . you just have to look to higher order divergences, and find out this never stops . . . but you have to show that you don't keep getting more and more in trouble. Which I'll show you why . . . in a minute.*

To make this even clearer, if I have to; to look at it another way. We originally have to do an integral that looks like this. Then there's another term, which I'll write as a factor $\bar{\psi}$ *gradient dagger*[79] *minus A, which I will write like this . . . You integrate this over* ψ *and also over A,*

$$\begin{aligned} &\int \mathcal{D}A_\mu \, e^{i \frac{1}{g_0^2} \int F_{\mu\nu} F_{\mu\nu}} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \, e^{i \int \bar{\psi}(i\not{D} - m)\psi} \\ &= \int \mathcal{D}A_\mu \, e^{i \frac{1}{g_0^2} \int F_{\mu\nu} F_{\mu\nu}} \underbrace{\det\left(i\not{D} - m\right)}_{g(A) \sim e^{i \ln \Lambda^2 f(A)}} \end{aligned} \tag{C.17}$$

That's the kind of thing we're trying to do. Now we can imagine first that I had just done this integral [over ψ] *completely, it would be nice if we could do it for arbitrary A; this is some terrifying functional of A.* [question from student[80]] *Yes, yes, yes, exactly. This is one over—no,*

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[79] RPF habitually says "dagger" to mean what we call "slash"

[80] Evidently RPF wrote $g(A)$ first without identifying it as the functional determinant.

the determinant of the Dirac operator i gradient minus m minus A, with some color terms; yes, that's just what it is; it's the same thing, we can't work this out either. Now we can expand this by perturbation theory, and try to make a calculation, and we discover that this is . . . at very short distances, high momentum, at very short distances there's some trouble. The trouble comes from too many gradients on top of each other, the propagators from this thing have delta functions in the origin, delta functions on a line, two of them on top of each other, they strongly diverge. So for very high frequency A, this function has to be discussed, this is a little bit wrong. So we had to fix it a little bit. It's still a determinant, we just fixed it by a cutoff. When we fix it by the cutoff, we discover that this thing is e to the i times $\ln \Lambda^2$ *multiplied by another function of A, of course you can always write it that way. But insofar that this only involves high frequencies, short distances, A is at two points very close together.*

And, if everything has been done right, since this expression here [the determinant] *is gauge invariant, with respect to A, because if I make a gauge transformation of A, and then fix up the* ψ *so as to get the same answer, this has to be a gauge invariant expression . . . we discovered over there that it involved the gradient of A squared. But we know that it's gauge invariant, and therefore if I did it completely, I would get the whole string* [the three terms in F^2]*, I could only get this; this is the only gauge invariant expression which starts like that. And that it starts like that is a statement of the forms that we got—by the way, although we didn't notice it, but this A, this kind of propagator, is just—see, if I put an A on each side of this, this becomes* A_ν *squared and two gradients. Well that's* ∂_μ *squared. Let's figure out which way* [to contract the Lorentz indices]*. The other one is* $q_\mu A_\nu q_\nu A_\mu$ *. . . Then assume an A which is a plane wave and substitute it in this expression* [the gluon kinetic term]*; you would get this kind of thing back* $[\delta_{\mu\nu} q^2 - q_\mu q_\nu]$*, so this is in fact the operation producing this kind of combination. So we're getting the first term right. Because we only looked at the two-gluon. If we looked at the three-gluon, we would be* **surprised** *to discover that it produces A dot A to the cube* $[A_\nu^\times A_\mu^\cdot \partial_\mu A_\nu]$*, which is just this combination. And even* **more** *surprised to discover that the coefficient is the same* g_0 *exactly. The surprise would disappear when we realized that it has to be with a cutoff scheme that* [preserved gauge invariance] *. . .*

And in the same way, when we go to integrate over A, we know that there's a problem with the meaning of this thing [the functional measure]*; let's forget about it. But we could imagine some kind of rule: stop the integrations above a certain high frequency—unfortunately, that's not gauge invariant—do a lattice. Well let's say stop high frequencies, forget about . . . one of these days—dimensional renormalization. So you cut this off at high frequency; you say wait a minute, what if I cut it off at a different high frequency? Then I could say that the intermediate between the medium high and the very high frequencies is what I'm integrating over to see what happens if I cut it off at the medium frequency. That will produce a number of terms that will involve the logarithm of the very high and the medium high frequencies. And the coefficient of that log . . . will again have to be gauge invariant and have the same kind of form . . .* [change of tape]

. . . it's just pretty; the only place where there appear to be divergences are just the places you need to make the simple form to be the same shape as the original one. Sometimes the good way to look at it is that this thing can be compensated by putting a term like this with a g^2*. The Lagrangian* [gets a] *correction term, this can be put in by putting a term like that times some q times some number. So the Lagrangian has a correction term, those are called counterterms; in other words, if we started with a Lagrangian, instead of saying* $1/g_0^2$ *exactly, times* $F_{\mu\nu} F_{\mu\nu}$*, we say we're going to with a Lagrangian which has already in it counterterms, this thing minus those numbers, which you're gonna find out what they're gonna be, times* $F_{\mu\nu} F_{\mu\nu}$*, which are counterterms. That's the Lagrangian that I started with, and then we're going to have a cutoff at* Λ*. And the cutoff at* Λ *is going to be equivalent to making corrections—divergences—well, they're not divergent because we're cutting them off, which undo these, to get something which is independent of* Λ*. So if I write the real g* [the renormalized value]

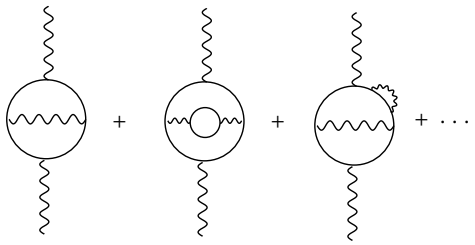
$$\frac{1}{g^2} = \frac{1}{g_0^2} + \delta\left(\frac{1}{g^2}\right) \tag{C.18}$$

this is equivalent to starting with some constant, call that g_0*, the constant we started with, plus some counterterm which will depend on* Λ*, and they're built in such a way as to compensate the divergences that we get here. The net result when we're all finished is just that constant times* $F_{\mu\nu} F_{\mu\nu}$*, independent of the cutoff. And that's where all those formulas come from, that talk about the* $1/g^2$ *being corrected by things that depend on* Λ*. Alright? Any other questions? It's just another way of describing the same thing. But the beauty of it is . . . the divergent terms are exactly right to reproduce the form of the Lagrangian, and therefore by changing the coupling constant we can undo the* Λ *dependence. Alright?*

Now I do have to complete the discussion, to discuss higher order calculations . . . I only did this one [the gluon vacuum polarization] *to lowest order—I didn't do everything, I didn't do these loops, but let's suppose I had; well let's say this one, I don't care, look, it doesn't make any difference, quarks or gluons* [in the loop]*. I'm not going to worry about . . . However, we just noticed when we were counting divergences that now we're in trouble, because now after we did all that, then we look and we find something like this, and that table of calculations says that this will have the same divergence as this and this.*

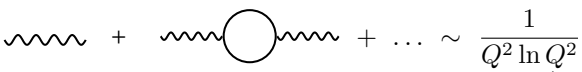
By the way, there was a step in here that was very clever: taking this and putting it [in the denominator of (C.14)] *implies something about the higher orders—the leading logs, what that's all about, is that I have not only*

taken this [one loop] *diagram,*



but I've added this diagram, and this diagram [two loops] *and so on, to get the sum of one plus x plus* x^2 *plus* x^3 *plus* x^4 *... is equivalent to* $1/(1-x)$*, and when I did this* [(C.14)]*, I was already predicting the higher terms, but I know where they're coming from, obviously ... So I've done all the single loops.*

Now, so this looks as if it's going to produce another contribution to the log, and so on, and the millions of diagrams—but it's not true. Because, I look at it this way, every time from now on that I see a gluon propagator, I really should correct it, or could—can correct it, by putting a loop in there like so, or two loops or three loops, and so on,



$$+ \ \ldots \sim \frac{1}{Q^2 \ln Q^2} \tag{C.19}$$

I want to include all these loops. So what this ought to be, this line now really means the propagation compensated, or corrected by these loops. Now we found out that—I guess I should have emphasized, that when I make this choice, substitute that back in here, I find this thing, β *times the log of* Λ^2*, minus this logarithm; this is the logarithm of* Q^2 *... And therefore the effective propagation is not really* $1/Q^2$*, but is really* $1/(Q^2 \ln Q^2)$ *if the loops are included. That's slightly more convergent. That goes down a little bit faster than* Q^2*. So that these propagators are not strictly speaking* $1/Q^2$*, they're* $1/(Q^2 \ln Q^2)$*. By the way, we sometimes write that as* $\alpha(Q^2)/Q^2$*; we talked about that. Anyway it's one over log, so these divergences are no longer computed right by just saying things like* $\int d^4p/p^4$ *with zero powers left over, which is equivalent of course to* $\int d(p^2)/p^2$*. But the propagators are not* $1/p^2$*, they have logs in them, and I have to tell you how many there are; the lowest possibility is that it begins with* $\ln p^2$*,*

$$\int^{\Lambda} \frac{dp\, p}{p^2 \ln p^2} \sim \ln(\ln \Lambda^2) \tag{C.20}$$

Indeed, there are terms, you might have higher powers of logs ... but not worse, you have at least the log. But you see what this looks like, this differential log over log? This is log log, less divergent. So if I integrate this to some high frequency, this is still divergent, but it's $\ln(\ln \Lambda^2)$*. And when I go to the next order, I'm going to get log log. Of course there can be terms with log squared here, but those will be convergent:*

$$\int^{\Lambda} \frac{dp\, p}{p^2 \ln^2 p^2} \sim \int^{\Lambda} \frac{d \ln p^2}{\ln^2 p^2} \sim \frac{1}{\ln \Lambda^2} \tag{C.21}$$

In other words this integral [(C.20)]*, this doesn't quite converge, but if I had more logs in there it would converge. So that there are little log-log divergences, no worse.*

So the next order terms, when you include the corrections in them from the lower order, does not produce the same divergence ... You might say, well how do I know that there are ... maybe they're only log squared or log cubed ... β_1 *terms ...* [Question from student: why did you have just two powers of the momenta in the numerator of that integral?] *Well it's a logarithmic divergence. I don't care about ... it's just to understand the log ... The differential log over log is log log, that's it.*

Now, you say well now I'm going to go on to the third. But it's no longer true that it goes on to the next one, because the higher ones give more powers of logs down here. And more powers of logs down here, it converges, it stops. Thank God. Now you say, well, it wasn't really log p^2*, it was that plus* $\ln \ln p^2$*, and I'll let you make the argument that this doesn't change the divergence; it doesn't make it converge any better ...* [Question from student about absence of triple logs] *w Because the next order produces an integral like this* [(C.21)]*... There's no way to isolate log-log. This correction is this propagator, which is corrected by one over log. If it's got a log-log in it, it's additional. In other words the corrections of* β_0 *log plus* β_1 *log log. So the correct thing to put in there, if it's anything, is* $\ln p^2$ *plus* $\ln \ln p^2$ *at that order. But that argument, that doesn't make any difference compared to the* $\ln p^2$ *when you calculated the divergence. What happens in the higher orders, you get more logs down here, but you get this to a higher power ... there's no way to isolate this ...*

[Question from student] *This has to do with the behavior in terms of* Q^2*. The propagator is* $\ln Q^2$ *... it's true that the correct formula has* $\ln Q^2$ *plus* $\ln \ln$ *terms; that doesn't ... you just get rid of this log and subtract it from the ... and try to get the log log isolated; there's nothing that comes out ... as you would like it, as you might imagine it. ...* $Q^2(\ln Q^2 + \ln \ln Q^2)$*. Of course there's no propagator that goes inversely to log log. That's not the way it goes. It goes inversely to the sum. And that would be log log log. You don't get this kind of a form, because you can't isolate that piece ... My argument here is very heuristic, but it does work and it gives you an idea of why ... this is the way the thing works out ... more or less why it works out ... any other question?*

In order to add to your confusion ... different conventions ... different ways of looking at it; some are better than others. If you want to read the literature, you have to read everybody's ideas. Some are better ... because they get rid of some confusion, so they straighten something out. Now if you learned only the way which is all straightened out ... then you have some trouble reading literature in which something is a little older or something which the guy is using some old-fashioned idea; well I wouldn't say old-fashioned but less ... then you couldn't understand the subject completely. What I described, I had tried to prove differently than is general

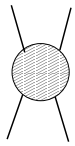
in any textbook—if it's in some textbook, you don't think it's original, not at all . . . I did it myself—but I have to tell you about what I consider a kind of mistake, okay? Which is very prevalent and it's all over the place. It's **not** *the way I'm trying to explain it. Now the way I'm trying to explain it is that the coupling constant—when you make a cutoff, you change the theory. And the answers in general appear to depend on the cutoff. But it turns out they depend on two things: the way you cut it off, Λ, and also the coupling constant you put in. But by the very wonderful situation we have of renormalization, that when you adjust the coupling constant correctly, when you change the cutoff, you change the coupling constant, you'll get the same predictions in the long-range wavelength physics . . . And that's what I wrote in the beginning, was a formula for how you have to change . . . in order to make the results independent of the cutoff. You change the cutoff, you'd better change the* [bare coupling] *. . .*

Next. It turned out that in many circumstances where you could expect the mass of the quarks to be unimportant . . . that for such processes, the behavior of the process could be worked out as a perturbation theory in g,

$$\begin{aligned} & c_1\pi g_0^2 + g_0^4\left(b_1 \ln\frac{\Lambda^2}{Q^2} + a_1\right) + \dots \\ = \; & c_1\alpha(Q^2) + a_1\alpha^2(Q^2) + \dots \end{aligned} \tag{C.22}$$

These coefficients depend on the logarithm of the cutoff and the momentum of the operation, the process, some momentum associated with the process, some definition of the momentum. And this sum, these logs, could be summed so this was written in the form of $\alpha(Q^2)$ plus $a_1\alpha^2(Q^2)$ and so on, and this is for some physical process. In other words, for some physical process, the calculations go like this. For a different process, all the coefficients would be different . . . there might be a π, so let's put one in here[81] *. . . anyhow, it would go like this and it would be—so, therefore when you start to work perturbation theory there's a rule . . . work at second order you get the log, but you've already eaten that when you made the substitution. So this is a way I managed to write it, and it's perfectly okay, and the formula for how loops . . . describe . . . the type of thing that's connected . . . which we wrote down . . . I'm just repeating . . . thing I found out by . . . or rather not the best way to do it.*

Now let me tell you the wrong way, what I consider not as good a way. It's a way that you could have done it, but it's got annoyances in it. It works like this. You start out to define an $\underline{\alpha}$. Now in order to make sure that this $\underline{\alpha}$ is not exactly the same as that α, I'm going to put an underline on it, so you'll always know which one I'm talking about. Now define an $\underline{\alpha}(Q^2)$ **by** *a physical process. I'm going to give you examples . . . For instance, we recalculate the scattering of two quarks, to* **all** *orders, exactly. All orders exactly, it's going to be written as this super-duper $\underline{\alpha}$ over Q^2,*



$$\equiv \frac{\underline{\alpha}(Q^2)}{Q^2} \tag{C.23}$$

and that's going to define $\underline{\alpha}$. There would be no such thing as finding a power series expansion of the coefficient, which is what I would do there [eq. (C.22)]*; . . . that's alpha, by definition, to all orders. Another thing would be, another way for example, when we talk about $e^+e^- \to$ hadrons, the ratio $[R]$, and remember that the formula for that ratio in the first order perturbation theory is $1 + \alpha/\pi$, remember that? Or rather $1 + g^2/\pi$? I'm not worried about the 4π's in the definition of $\alpha(g^2)$, that's a pain in the ass that I can't remember. Let me* **define** *$\underline{\underline{\alpha}}(Q^2)$; this is another definition, let's put two lines, it's another definition, it's identically equal,*

$$\underline{\underline{\alpha}}(Q^2) = \pi(R-1) \tag{C.24}$$

But anyway, we could do this to all orders, this is defined to all orders. So there's no such thing as a perturbation expansion for this to first order and next order and next order . . . it's just a definition. That would be a possibility. As it turns out, that up to the first two orders, the formula for this $\underline{\alpha}$ and the formula for that $\underline{\underline{\alpha}}$ and my α all agree, in terms of the first log and the log-log. All of those formulas satisfy

$$\frac{d\alpha}{d\ln Q^2} = \frac{\beta_0\alpha^2}{1-\frac{\beta_1}{\beta_0}\alpha^2} = \beta_0\alpha^2 + \beta_1\alpha^4 + \frac{\beta_1}{\beta_0}\alpha^6 + \dots \tag{C.25}$$

And for every one of these definitions, for any one of these alpha bar things, the $\underline{\alpha}$ satisfies exactly the same equation, up to the fourth order,

$$\frac{d\underline{\alpha}}{d\ln Q^2} = \beta_0\underline{\alpha}^2 + \beta_1\underline{\alpha}^4 + \beta_2\underline{\alpha}^6 + \dots \tag{C.26}$$

but the next order depends on the process. These [the coefficients of the first two terms] *don't. In other words it depends on the process needed to define . . . the $\underline{\alpha}$. So for practical purposes up to second order it doesn't make any difference, but if you want to make things definite so that one guy can compare his results to the other, in higher order, they're all mixed up, because one guy is using one way, another is using another way, because of the difference in processes. You say what's any better, why not use a definite process? Because . . . Instead of using a definite process, I used a definite theory . . . You'll notice this $[\beta_2 = \beta_1^2/\beta_0]$ is a special choice, but it's definite, and this involves . . . independent of process. And an advantage is, you don't have to compute this special process . . . you've got it done . . . If you want to know some physics then you have to compute. And that's saying that you should really*

[81] I have the π in (C.22) crossed out in my notes

calculate the power series for this process, in terms of α, my α ... We should naturally expect to do each process separately as a perturbation expansion, instead of arbitrarily choosing one ... one is no better than the other ... it all adds confusion to ...

They then said that the physical coupling constant depends on Q^2, but there's no definition, it depends on how you define it. You could say that the coupling constant depends on momentum squared, but ... So, people talk about this as if it's a running coupling constant, but you can't put that into the Lagrangian, as a running coupling constant. The only thing you can put into the Lagrangian is something that depends on Λ, not on Q, so I was rather confused ... You see how much confusion ... in the definitions; Politzer's λ ... we saw the equations depend on which method of cutoff you use, and how you define the constant, is it zero or is it Euler's constant times the log of 4π ... And on top of that, on top, I wanted to add the ambiguities that slipped in ... to define alpha and it's not ...

Now in the electrodynamic world, there was a wonderful special process to find the electric charge, which was unique, which is, let me evaluate the interaction of the particles when they're very far apart—the very long wavelength coupling of photons to electrons. In quantum chromodynamics you can't find any ... like that ... For very long wavelengths ... so we have no simple phenomenon which ... Any other questions? Alright then.

Someone asked me last time how dimensional renormalization produces the same results. The answer is more or less the following. You would make a process in the scattering—first of all we have less than four dimensions. We have something like

$$\frac{1}{g^2}\underbrace{\int F_{\mu\nu}^2 d^D x}_{\text{dimensions of Energy}^{4-D}} \tag{C.27}$$

and then integrate with respect to D dimensions of spacetime, D is not four. Now that means—and what about the dimensions? In $F_{\mu\nu}$ as you all know and must have written ... there's the combination of ∂A and A A, and that means that A is an inverse length or an energy. That's independent of dimension. And that $F_{\mu\nu}$ is an energy squared. And $F_{\mu\nu}$ squared is an energy to the fourth. A length is an inverse energy. So this quantity would have dimensions of energy to the $4-D$. Therefore g^2 is not dimensionless. g^2 has dimensions of energy to the $4-D$,

$$[g^2] = \text{Energy}^{4-D} = E^\epsilon \tag{C.28}$$

So here, one way is to just say, alright, I know that. Another way is to write g as some other constant times some particular length to the $4-D$, I'll call that epsilon:

$$g^2 = c_0 \lambda_P^\epsilon \tag{C.29}$$

... Now what happens is, if you do perturbation theory, g^2 plus g^4 times an integral, same way as we did before. Except those integrals because they don't have d^4p any more, they only have a D integral, are less divergent, in fact they converge. So you can actually do the integral, and there's no problem, and you find that the integral varies as $Q^{-\epsilon}$

$$\begin{aligned} g^2 + g^4 Q^{-\epsilon}\left(\frac{2\beta_0}{\epsilon}\right) &= \frac{1}{\frac{1}{g^2} - \frac{2\beta_0 Q^{-\epsilon}}{\epsilon}} \\ &\approx \frac{1}{\frac{2\beta_0}{\epsilon}\lambda_P^{-\epsilon} - \frac{2\beta_0}{\epsilon}Q^{-\epsilon}} \end{aligned} \tag{C.30}$$

minus ϵ because it's $D-4$ ... so you get this kind of a term and ... it's no problem, everything converges and it's fine. But as we vary ϵ, we discover that the coefficient that we actually get, the coefficient here, is a certain constant $[2\beta_0]$ over ϵ. That is, if I did the calculation with different ϵ, I'd get something that varies with ϵ this way: the coefficient diverges as $\epsilon \to 0$.

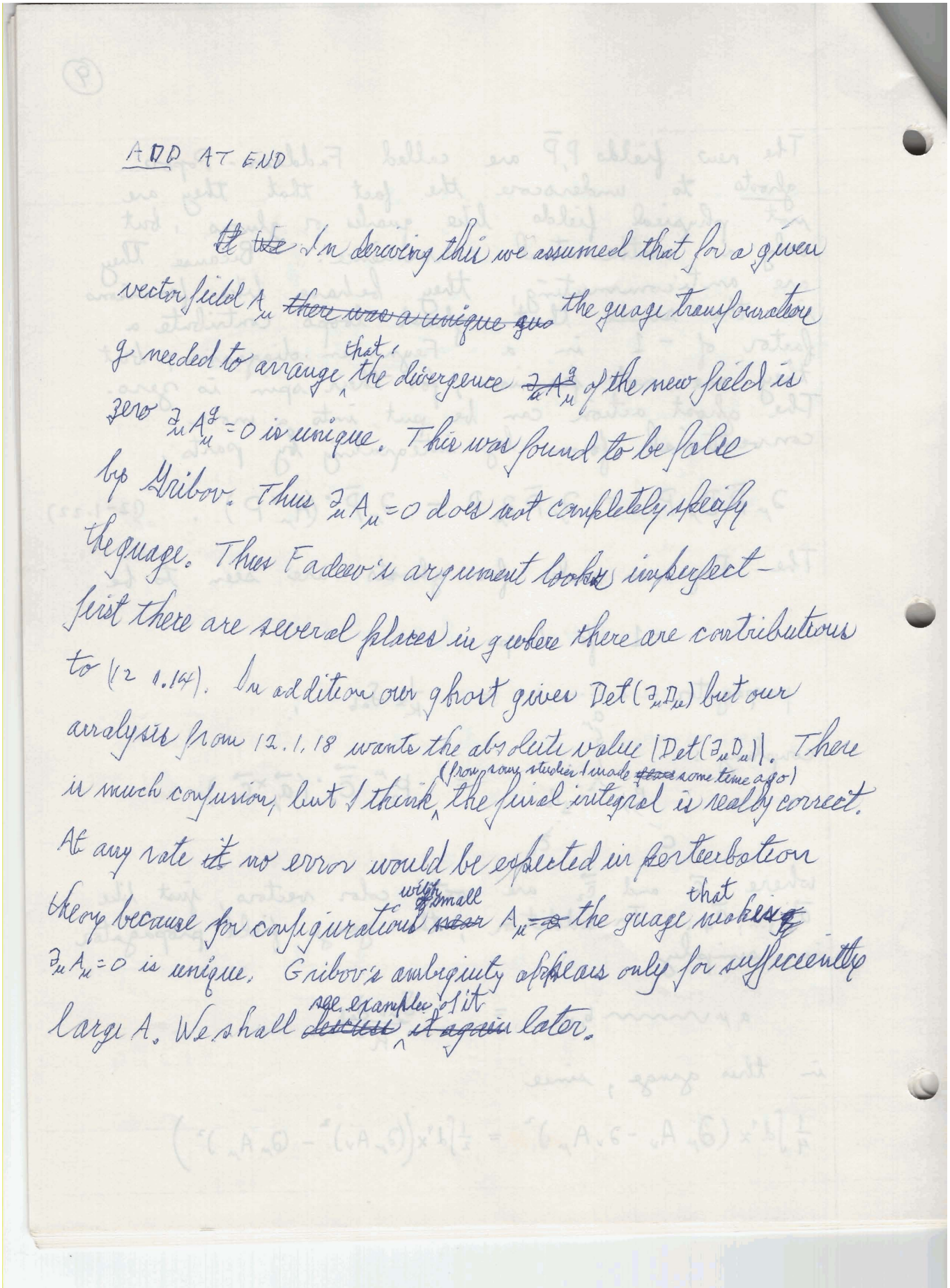
I can go through all this usual stuff of rearranging the sum and ... when Q is very large this is small; but the point is that aside from constants which will be taken out [RPF writes out the right-hand side of eq. (C.30) at this point] *... and as ϵ approaches zero,*

$$2(\lambda_P^{-\epsilon} - Q^{-\epsilon}) = -\epsilon \ln(\lambda_P^2/Q^2) \tag{C.31}$$

The point is, the theory is convergent. It depends on ϵ, and has a very high coefficient as ϵ goes to zero, and in the limit produces logarithms, just like the logarithms you see, and we have to adjust the coupling constant. So it has the right behavior with ϵ ... the same problem, how does this g depend on ϵ? [Student: in dimensional regularization, don't you want to take λ to be close to the energy scale of the ...] λ_P *... In the end yes, it's better, yes I think that's the right thing to do ... So what we're saying is that the coupling constant has the dimensions of the physical energies we're interested in ... but it turns out the strength varies inversely as ϵ ...*[82]

Appendix D: Revision examples

Additions written by RPF to the revision of lecture 12.



ADD AT END

~~It was~~ In deriving this we assumed that for a given vector field A_μ ~~there was a unique g~~ the guage transformation g needed to arrange that the divergence $\partial_\mu A^g_\mu$ of the new field is zero $\partial_\mu A^g_\mu = 0$ is unique. This was found to be false by Gribov. Thus $\partial_\mu A_\mu = 0$ does not completely specify the guage. Thus Fadeev's argument looks imperfect - first there are several places in g where there are contributions to (12.1.14). In addition our ghost gives $Det(\partial_\mu D_\mu)$ but our analysis from 12.1.18 wants the absolute value $|Det(\partial_\mu D_\mu)|$. There is much confusion, but I think (from some studies I made some time ago) the final integral is really correct. At any rate no error would be expected in perturbation theory because for configurations with A_μ small the guage that makes $\partial_\mu A_\mu = 0$ is unique. Gribov's ambiguity appears only for sufficiently large A. We shall see examples of it later.

Additions written by RPF to the revision of lecture 13.

(7)

like a cross product of the vectors (Q_1, Q_2) and (P_1, P_2).

Now we would like to deduce the potential between quarks from the operator $(\mathbb{D}\cdot\nabla)^{-1}\nabla^2(\mathbb{D}\cdot\nabla)^{-1}$ in 12-3.22. Although $\mathbb{D}\cdot\nabla$ cannot be inverted in closed form, it can be expanded in powers of the gluon field, which corresponds to weak coupling. We get

$$\frac{1}{\mathbb{D}\cdot\nabla}\nabla^2\frac{1}{\mathbb{D}\cdot\nabla} = \frac{1}{\nabla^2} + 2\frac{1}{\nabla^2}A^x\cdot\nabla\frac{1}{\nabla^2} + 3\frac{1}{\nabla^2}(A^x\cdot\nabla)\frac{1}{\nabla^2}(A^x\cdot\nabla)\frac{1}{\nabla^2} + \cdots \qquad (12\text{-}3.24)$$

The first term corresponds to the Coulomb potential, as in (12-3.13).

If the following terms are evaluated to 2nd order we get a strong attractor. In $\beta_0 = (11-\frac{2}{3}n_f)$ it contributes +13 units - most of the confining effect. Thus the instantaneous coulomb force probably rises with distance. The charge density (and current) of transverse (real) gluons and quarks are $(\Gamma + P^x A)$ and their interaction makes via $\frac{1}{\mathbb{D}\cdot\nabla} = \frac{1}{\nabla}$ with a vacuum polarization of the normal sign, contributions to β_0 of $-1-\frac{2}{3}n_f$.

To second order in the

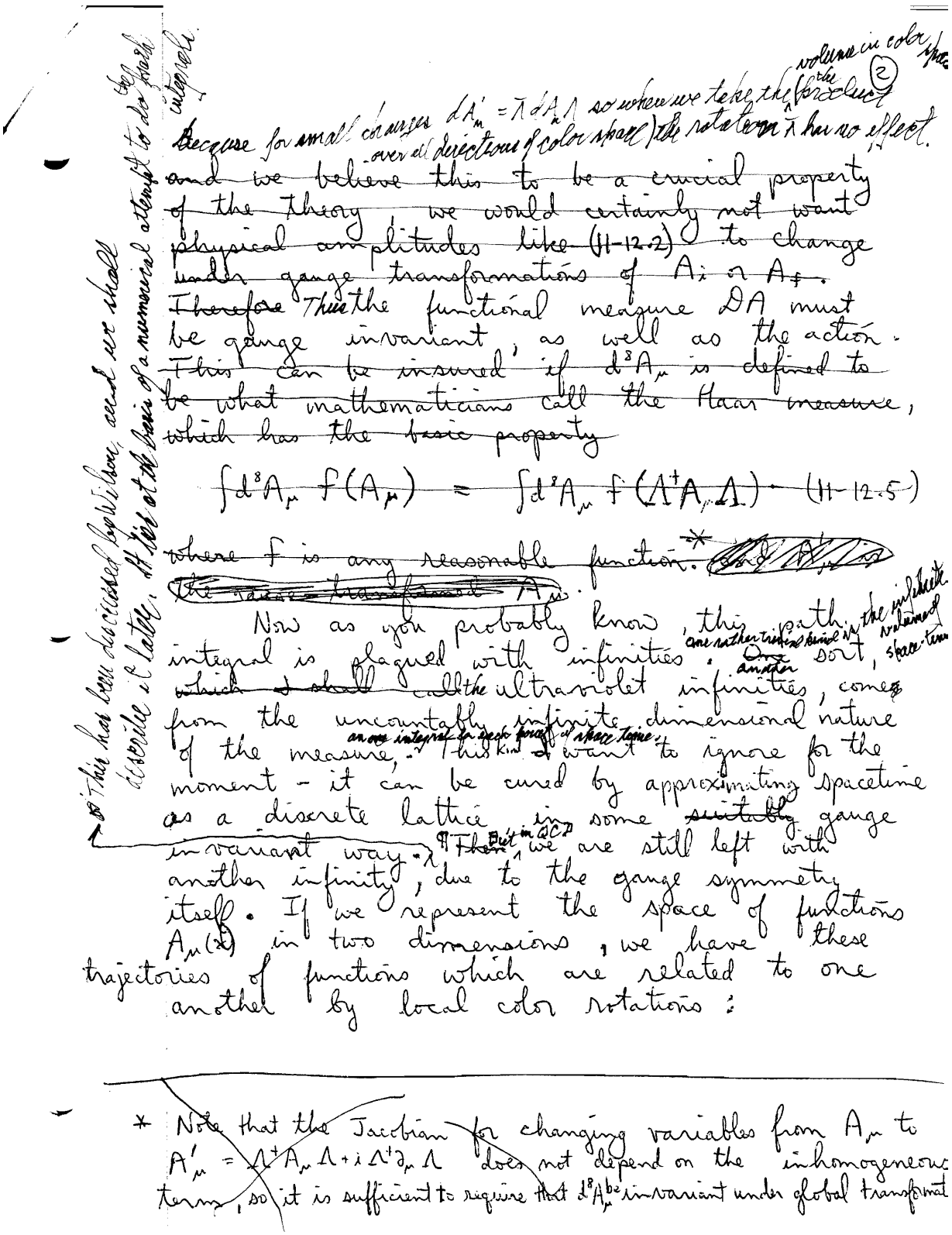
This leads to our interesting model of a string connecting heavy quarks. Because of the rapid rise (with R) of the force between charges, unbalanced color charges at any distance is intolerable. Suppose we start with a red quark at 0 (and say another far away to the right. By creating a dipole gluon $R\bar{G}$ within this range where opposite colors are tolerable we cancel redness at larger transverse distances. But then the G end of the gluon is

unbalanced, so another gluon dipole $\bar{G}B$ forms (the energy for which coming from the decrease in G energy). This continues until we reach the final quark. Thus we have a state of superpositions of color arrangements $\bar{R}G$ = gluon, ○ quark.

↓ Note out here all the colors cancel and there is not much energy in the Coulomb-like quantum chromostatic field

We shall have a more precise description of these strings later on in the course and to see if this is a useful viewpoint of a string. (Such pictures will have also been discussed by Greenstreet)

There were sometimes also subtractions: edits by RPF to the revision of lecture 8.



Appendix E: Hadron masses and quark wave functions

The following three pages were copied out of an unidentified textbook and handed out at the beginning of the course. Handwritten corrections of quark wave functions were added by me.

156 HADRON MASSES AND HEAVY QUARKS

(XVII.i)

	Mass(MeV)	SU(3)	Spin	Ispin	T_3	Y	c
P	938.3	8	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0
N	939.6	8	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	0
Λ	1115.6	8	$\frac{1}{2}$	0	0	0	0
Σ^+	1189.4	8	$\frac{1}{2}$	1	1	0	0
Σ^0	1192.5	8	$\frac{1}{2}$	1	0	0	0
Σ^-	1197.3	8	$\frac{1}{2}$	1	−1	0	0
Ξ^0	1314	8	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	−1	0
Ξ^-	1321	8	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	−1	0
Δ^{++}		10	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	1	0
Δ^+	1230	10	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	1	0
Δ^0	1234	10	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	1	0
Δ^-		10	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	1	0
Σ^{*+}	1382	10	$\frac{3}{2}$	1	1	0	0
Σ^{*0}	1382	10	$\frac{3}{2}$	1	0	0	0
Σ^{*-}	1387	10	$\frac{3}{2}$	1	−1	0	0
Ξ^{*0}	1532	10	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	−1	0
Ξ^{*-}	1535	10	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	−1	0
Ω^-	1672	10	$\frac{3}{2}$	0	0	−2	0
Λ_c^+	2273	$\bar{3}$	$\frac{1}{2}$	0	0	$\frac{2}{3}$	1
-	?	$\bar{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{3}$	1
-	?	$\bar{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$	1

(XVII.1) cont.

	quark wave function (+ cyclic permutations)
p	$\|uud\rangle(2\|{++-}\rangle - \|{+-+}\rangle - \|{-++}\rangle)/3\sqrt{2}$
N	$-\|ddu\rangle(2\|{++-}\rangle - \|{+-+}\rangle - \|{-++}\rangle)/3\sqrt{2}$
Λ	$(\|uds\rangle - \|dus\rangle)(\|{+-+}\rangle - \|{-++}\rangle)/2\sqrt{3}$
Σ^+	$\|uus\rangle(2\|{++-}\rangle - \|{+-+}\rangle - \|{-++}\rangle)/3\sqrt{2}$
Σ^0	$(\|uds\rangle + \|dus\rangle)(2\|{++-}\rangle - \|{+-+}\rangle - \|{-++}\rangle)/6$
Σ^-	$\|dds\rangle(2\|{++-}\rangle - \|{+-+}\rangle - \|{-++}\rangle)/3\sqrt{2}$
Ξ^0	$\|uss\rangle(\|{++-}\rangle + \|{+-+}\rangle - 2\|{-++}\rangle)/3\sqrt{2}$
Ξ^-	$\|dss\rangle(\|{++-}\rangle + \|{+-+}\rangle - 2\|{-++}\rangle)/3\sqrt{2}$
Δ^{++}	$\|uuu\rangle\|{+++}\rangle$ (no permutations)
Δ^+	$\|uud\rangle\|{+++}\rangle/\sqrt{3}$
Δ^0	$\|udd\rangle\|{+++}\rangle/\sqrt{3}$
Δ^-	$\|ddd\rangle\|{+++}\rangle$ (no permutations)
Σ^{*+}	$\|uus\rangle\|{+++}\rangle/\sqrt{3}$
Σ^{*0}	$(\|uds\rangle + dus\rangle)\|{+++}\rangle/\sqrt{6}$
Σ^{*-}	$\|dds\rangle\|{+++}\rangle/\sqrt{3}$
Ξ^{*0}	$\|uss\rangle\|{+++}\rangle/\sqrt{3}$
Ξ^{*-}	$\|dss\rangle\|{+++}\rangle/\sqrt{3}$
Ω^-	$\|sss\rangle\|{+++}\rangle$ (no permutations)
Λ_c^+	$(\|udc\rangle - \|duc\rangle)(\|{+-+}\rangle - \|{-++}\rangle)/2\sqrt{3}$
-	$(\|usc\rangle - \|suc\rangle)(\|{+-+}\rangle - \|{-++}\rangle)/2\sqrt{3}$
-	$(\|dsc\rangle - \|sdc\rangle)(\|{+-+}\rangle - \|{-++}\rangle)/2\sqrt{3}$

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(XVII.ii)

	MASS	SU(3)	SPIN	ISPIN	T_3	Y	C	quark wave function
$\pi^{\pm}$	139.6	8	0	1	± 1	0	0	$u\bar{d}(d\bar{u})$
π^0	135.0	8	0	1	0	0	0	$(u\bar{u}-d\bar{d})/\sqrt{2}$
$K^{\pm}$	493.7	8	0	$\frac{1}{2}$	$\pm\frac{1}{2}$	± 1	0	$u\bar{s}(s\bar{u})$
$K^0(\overline{K^0})$	497.7	8	0	$\frac{1}{2}$	$\mp\frac{1}{2}$	± 1	0	$d\bar{s}(s\bar{d})$
η	548	8	0	0	0	0	0	$(2s\bar{s}-u\bar{u}-d\bar{d})/\sqrt{6}$
η'	958	1	0	0	0	0	0	$(u\bar{u}+d\bar{d}+s\bar{s})/\sqrt{3}$
$\rho^{\pm}$	776	8	1	1	± 1	0	0	$u\bar{d}(\bar{d}u)$
ρ^0		8	1	1	0	0	0	~~$u\bar{d}(\bar{d}u)$~~ $(u\bar{u}-d\bar{d})/\sqrt{2}$
ω	782	1⊕8	1	0	0	0	0	$(u\bar{u}+d\bar{d})/\sqrt{2}$
$K^{*\pm}$	892	8	0	$\frac{1}{2}$	$\pm\frac{1}{2}$	± 1	0	$u\bar{s}(s\bar{u})$
$K^{*0}(\overline{K^{*0}})$	899	8	1	$\frac{1}{2}$	$\mp\frac{1}{2}$	± 1	0	$d\bar{s}(s\bar{d})$
ϕ	1020	1⊕8	1	0	0	0	0	$s\bar{s}$
$D^{\pm}$	1868	$\bar{3}(3)$	0	$\frac{1}{2}$	$\pm\frac{1}{2}$	$\frac{1}{3}$	± 1	$c\bar{d}(d\bar{c})$
$D^0(\overline{D^0})$	1863	$\bar{3}(3)$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\mp\frac{1}{3}$	± 1	$c\bar{u}(u\bar{c})$
$F^{\pm}$	-	$\bar{3}(3)$	0	0	0	$\pm\frac{2}{3}$	± 1	$c\bar{s}(s\bar{c})$
$D^{*\pm}$	2009	$\bar{3}(3)$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\mp\frac{1}{3}$	± 1	$c\bar{d}(d\bar{c})$
$D^{*0}(\overline{D^{*0}})$	2006	$\bar{3}(3)$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\mp\frac{1}{3}$	± 1	$c\bar{u}(u\bar{c})$
η_c	2980	1	0	0	0	0	0	$c\bar{c}$
J/ψ	3097	1	1	0	0	0	0	$c\bar{c}$
Υ	9460	1	1	0	0	0	0	$b\bar{b}$

Appendix F: Tables of hadrons

These tables of meson and baryons were written by RPF.

MESONS: J^{PC} (1986)

	0^{-+}	1^{--}	0^{++}	1^{+-}	1^{++}
	0^{-+} ($S=0, L=0$)	$S=1, L=0$	$S=1, L=1$	$S=0, L=1$	$S=1, L=1$
$I=1$	$\pi(140).02$	$\rho(770).59$	$a_0(980).96$	$b(1235)\ 1.53$	$a_1(1270)\ 1.61$
$I=0$	$\eta(549).30$	$\omega(783).61$	$f_0(975).95$	$h(1190)\ 1.42$	$f(1285)\ 1.64$
$I=0$	$\eta'(958).92$	$\varphi(1020)\ 1.04$	$f(1300)\ 1.69$	h'	$f(1420)\ 2.02$
$I=1/2$	$K(492).24$	$K(892).80$	$K(1350)\ 1.82$?	$K(1280)\ 1.64$	$K(1400)\ 1.96$
	~~$\pi(1300)\ 1.69$?~~	$\rho(1600)\ 2.56$			
	$\eta(1400)\ 2.07$		$f(1590)\ 2.53$?		
		$\varphi(1680)\ 2.82$?			

	2^{-+}	2^{++}	2^{--}	3^{--}	4^{++}
	$S=0, L=2$	$S=1, L=1,3$	$S=1, L=2$	$S=1, L=2$	$S=1, L=3$
$I=1$	$\pi(1680)\ 2.82$	$a(1320)\ 1.74$		$\rho(1690)\ 2.86$	
$I=0$		$f(1270)\ 1.53$		$\omega(1670)\ 2.79$	$f(2030)\ 4.12$
$I=0$		$f(1525)\ 2.48$		$\varphi(1850)\ 3.42$	
$I=1/2$	$K(1770)\ 3.13$	$K(1430)\ 2.04$		$K(1780)\ 3.17$	$K(2060)\ 4.24$
		$f(1720)\ 2.86$			

? Means "Not a well established Resonance."

BARYONS J^P (1986)

NEGATIVE PARITY

$\frac{1}{2}^-$	$3/2^-$	$5/2^-$	$7/2^-$
1	1	8	8
$\Lambda(1405)1.97$	$\Lambda(1520)2.31$	$N(1675)2.81$	$N(2190)4.79$
		$\Lambda(1830)3.35$	
8	8	$\Sigma(1770)3.15$	1
$N(1535)2.36$	$N(1520)2.31$		$\Lambda(2100)4.41$
$\Lambda(1670)2.79$	$\Lambda(1690)2.86$	10	
	$\Sigma(1670)2.79$	$\Delta(1930)3.72$	
8	$\Xi(1820)3.32$		$9/2^-$
$N(1650)2.72$			$N(2250)5.06$
$[\Sigma(1750)3.06]^\dagger$	8		
$\Lambda(1800)3.24$	$N(1700)2.89$		$11/2^-$
10	10		$N(2600)6.76$
$\Delta(1620)2.62$	$\Delta(1700)2.89$		
$[\Sigma(1750)3.06]^\dagger$			
10	?		
$\Delta(1900)3.61$	$\Sigma(1940)3.76$		

† LOCATION OF $\Sigma(1750)$ IS UNCERTAIN

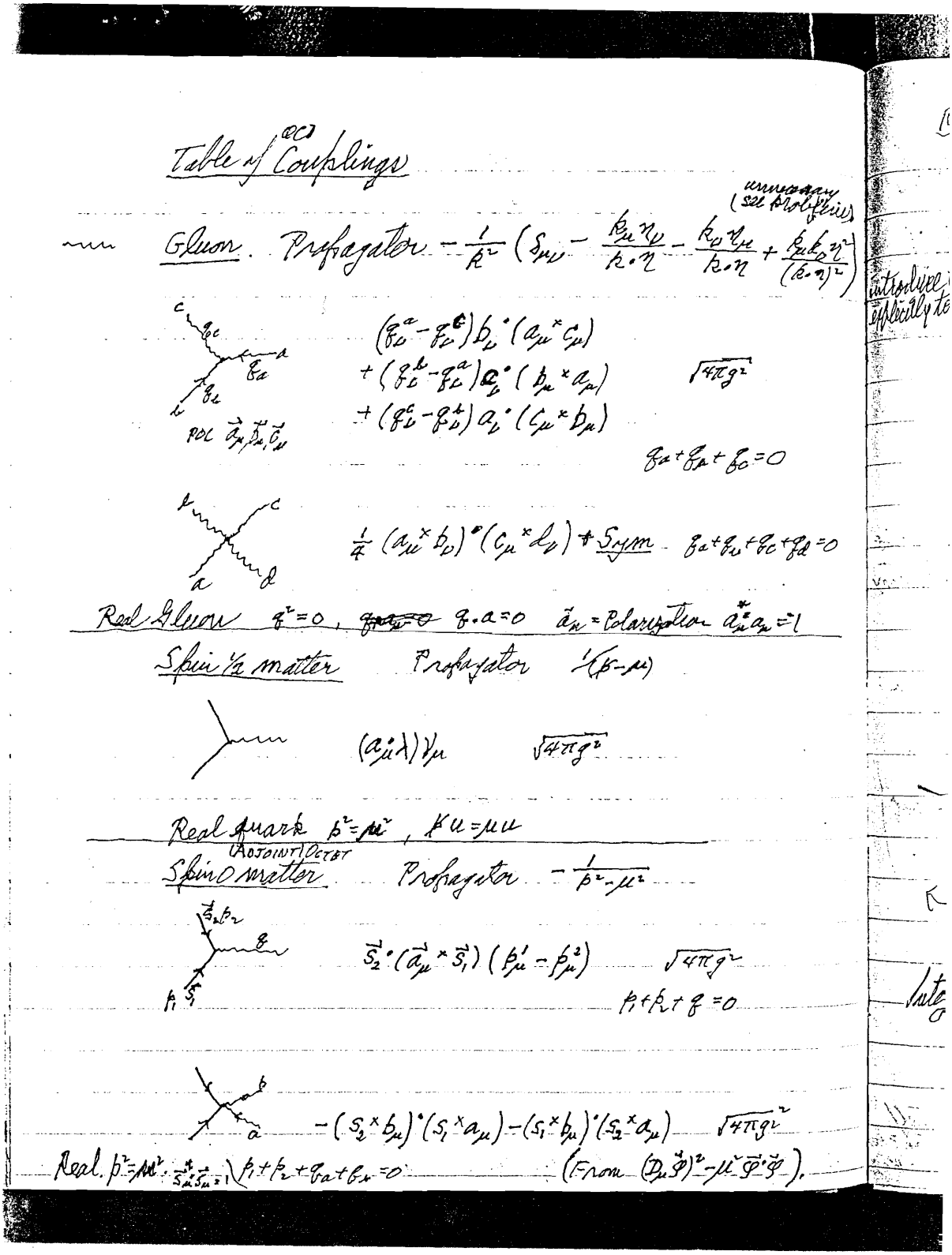
BARYONS J^P (1986)

POSITIVE PARITY

$\frac{1}{2}^+$	$\frac{3}{2}^+$	$\frac{5}{2}^+$	$\frac{7}{2}^+$
8	10	8	10
N(939) .88	Δ(1232) 1.52	N(1680) 2.82	Δ(1950) 3.80
Λ(1116) 1.25	Σ(1385) 1.92	Λ(1820) 3.31	Σ(2030) 4.12
Σ(1192) 1.42	Ξ(1533) 2.35	Σ(1915) 3.67	
Ξ(1318) 1.74	Ω(1672) 2.80		
		1	$\frac{9}{2}^+$
8	8	Λ(2110) 4.45	8
N(1440) 2.07	N(1720) 2.96		N(2220) 4.93
Λ(1600) 2.56	Λ(1890) 3.57	10	Λ(2350) 5.52
Σ(1660) 2.76		Δ(1905) 3.63	
	10		
8	Δ(1920) 3.69		$\frac{11}{2}^+$
N(1710) 2.92			10
			Δ(2420) 5.86
1			
Λ(1800) 3.24			
			UNKNOWN J^P
10			Σ(2250) 5.06 ?
Δ(1910) 3.65			[Ξ(1820) 3.32 $\frac{3}{2}$?]
			Ξ(2030) 4.12 ?

Appendix G: Rules for amplitudes and observables

These were also hand-written by RPF. Annotations in blue were made by me at the time the course was given.



$$\text{Rate} = \frac{(2\pi)^4 \delta^4(\Sigma p_{in} - \Sigma p_{out}) \left| T_i \right|^2 \prod_{\text{all out}} 2\pi \delta(p^2 - \mu^2) \frac{d^4p}{(2\pi)^4}}{\prod_{in} (2\epsilon_i)}$$

Rules for T:

Make diagrams at each junction p_μ is conserved.

Each quark $\frac{1}{\not{p} - \mu}$, Each gluon $\left(\delta_{\mu\nu} - \frac{p_\mu n_\nu}{p\cdot n} - \frac{p_\nu n_\mu}{p\cdot n}\right)$

Each coupling as previous page

Factor each loop 4-momentum integral $\therefore i \int \frac{d^4k}{(2\pi)^4}$

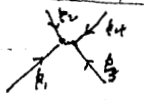
Factor another (-1) Spur for each quark loop.

Trace colors. Exchange rule -1 each exchange of $\not{}$

$$i \int \frac{d^4k}{(2\pi)^4} \frac{(1, k_\mu)}{(k^2 - 2p\cdot k - \Delta)^3} = \frac{(1, k_\mu)}{(p^2 + \Delta)} \cdot \frac{1}{32\pi^2}$$

$$\frac{1}{ab} = \int_0^1 \frac{dx}{ax + b(1-x)}$$

Two Body → Two Body



$(p_1 + p_3)^2 = s = (p_2 + p_4)^2$

$(p_1 + p_2)^2 = t = (p_3 + p_4)^2$

$(p_1 + p_4)^2 = u = (p_2 + p_3)^2$

$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$

$\text{RATE} = |T|^2 \cdot \frac{dt}{8\pi s^2}$

point

Rate $\frac{d\sigma}{dt} = \pi g^4 \frac{1}{(P_1+P_2)^2} |A|^2 / s^2$

Total cross section sum final spin & color, average initial spin and color. Values of $|A|^2$.

s = flavor Neglect masses. $s+t+u=0$

quark quark

(1) $Q_f Q_{f'} \to Q_f Q_{f'}$

quark antiquark

$Q_f \bar{Q}_{f'} \to Q_f \bar{Q}_{f'}$ $\frac{4}{9}\frac{s^2+u^2}{t^2}$

(different flavor)

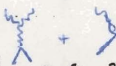
(2) $Q_f Q_f \to Q_f Q_f$ $\frac{4}{9}\left(\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2}\right) - \frac{8}{27}\frac{s^2}{ut}$ } exchange diagrams for same flavor quarks

(3) $Q_f \bar{Q}_f \to Q_f \bar{Q}_f$ (same flavor) $\frac{4}{9}\left(\frac{s^2+u^2}{t^2} + \frac{u^2+t^2}{s^2}\right) - \frac{8}{27}\frac{u^2}{st}$

(same as (2) with $s \leftrightarrow u$).

quark antiq to gluons

(4) $Q_f \bar{Q}_f \to GG$ $\frac{32}{27}\left(\frac{u^2+t^2}{ut}\right) - \frac{8}{3}\left(\frac{u^2+t^2}{s^2}\right)$



Relative $\frac{3^2}{6\cdot 6}$ in multiplicity in/multiplicity out

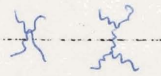
different average over initial states.

(5) ~~$Q_f G \to Q_f G$~~ $GG \to \bar{Q}_f Q_f$ $\frac{1}{6}\left(\frac{u^2+t^2}{ut}\right) - \frac{3}{8}\left(\frac{u^2+t^2}{s^2}\right)$

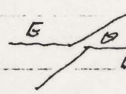
Reverse of $Q_f G \to Q_f G$ is one antiparticle in initial or final state.

(6) $Q_f G \to Q_f G$ $-\frac{4}{9}\left(\frac{u^2+s^2}{us}\right) + \left(\frac{u^2+s^2}{t^2}\right)$

(7) $GG \to GG$ $\frac{9}{2}\left(3 - \frac{ut}{s^2} - \frac{us}{t^2} - \frac{st}{u^2}\right)$



In C.M.



$s = 4E^2$ $t = -2E^2(1-\cos\theta)$ $u = -2E^2(1+\cos\theta)$

For small $t \sim 0$, $u = -s$: $QQ \to QQ$, $Q\bar{Q} \to Q\bar{Q}$ $\frac{8}{9}\frac{s^2}{t^2}$ | $Qg \to Qg$ $\frac{2}{3}\frac{s^2}{t^2}$ | $gg \to gg$ $\frac{9}{2}\frac{s^2}{t^2}$

quarks are $\frac{4}{9}$ as effective as gluons. $\therefore Q\frac{4}{9} + G$ Makes $Q\frac{4}{9} + G$ at $9/2\ s^2/t^2 = 18/(1-\cos\theta)^2$