Discussion Problems - Week 4

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Problem 1

Suppose that $X_1, \ldots, X_n$ are a random sample from the Normal distribution with unknown mean μ and unknown variance σ^2. Find the maximum likelihood estimator for μ and σ^2.

Solution

We are given a random sample $X_1, \ldots, X_n$ from a **Normal distribution**:

$$X_i \sim \mathcal{N}(\mu, \sigma^2), \quad i = 1, \ldots, n. \tag{1}$$

The probability density function (PDF) is:

$$f(x_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right). \tag{2}$$

Step 1: Write the Likelihood Function

Since the observations are independent, the **likelihood function** is:

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right). \tag{3}$$

Taking the **log-likelihood**:

$$\ell(\mu, \sigma^2) = \sum_{i=1}^{n} \left[-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(x_i - \mu)^2}{2\sigma^2}\right]. \tag{4}$$

Simplifying:

$$\ell(\mu, \sigma^2) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2. \tag{5}$$

Step 2: Find the MLE for μ

To maximize $\ell(\mu, \sigma^2)$, take the derivative with respect to μ and set it to zero:

$$\frac{\partial \ell}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu) = 0. \tag{6}$$

Solving for μ:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i. \tag{7}$$

Thus, the **MLE** for μ is the **sample mean**.

Step 3: Find the MLE for σ^2

Taking the derivative of $\ell(\mu, \sigma^2)$ with respect to σ^2:

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i - \mu)^2. \tag{8}$$

Setting it to zero:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2. \tag{9}$$

Thus, the **MLE** for σ^2 is the sample variance with denominator n.

Final Answer

Maximum Likelihood Estimators

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2. \tag{10}$$

These are the **MLEs for the mean and variance of a normal distribution**.

Problem 2

A random variable X is modeled using the following probability density function (PDF):

$$f(x \mid \theta) = \theta x^{-\theta-1}, \quad x > 1, \quad \theta > 0. \tag{11}$$

Suppose you observe the random sample $X_1, \ldots, X_n$. What is the maximum likelihood estimator (MLE) of θ?

Solution

We are given a **random sample** $X_1, \ldots, X_n$ from a distribution with the PDF:

$$f(x_i \mid \theta) = \theta x_i^{-\theta-1}, \quad x_i > 1. \tag{12}$$

Step 1: Write the Likelihood Function

Since the observations are independent, the **likelihood function** is:

$$L(\theta) = \prod_{i=1}^{n} \theta x_i^{-\theta-1}. \tag{13}$$

Rewriting:

$$L(\theta) = \theta^n \prod_{i=1}^{n} x_i^{-\theta-1}. \tag{14}$$

Step 2: Log-Likelihood Function

Taking the natural logarithm:

$$\ell(\theta) = n \ln \theta - (\theta + 1) \sum_{i=1}^{n} \ln x_i. \tag{15}$$

Step 3: Find the MLE for θ

To find the MLE, take the derivative of $\ell(\theta)$ with respect to θ and set it to zero:

$$\frac{d\ell}{d\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \ln x_i = 0. \tag{16}$$

Solving for θ:

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} \ln x_i}. \tag{17}$$

Final Answer

Maximum Likelihood Estimator

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} \ln x_i}. \tag{18}$$

Thus, the **MLE** for θ is the reciprocal of the sample mean of the logarithms of the observed values.

Problem 3

Suppose that $X_1, \dots, X_n$ form a random sample from the uniform distribution on the interval $[\theta_1, \theta_2]$, where both θ_1 and θ_2 are unknown ($-\infty < \theta_1 < \theta_2 < \infty$). Find the maximum likelihood estimator (MLE) of θ_1 and θ_2.

Solution

We are given a **random sample** $X_1, \dots, X_n$ from a **Uniform distribution**:

$$X_i \sim \mathcal{U}(\theta_1, \theta_2), \quad -\infty < \theta_1 < \theta_2 < \infty. \tag{19}$$

The probability density function (PDF) is:

$$f(x \mid \theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1}, \quad \theta_1 \leq x \leq \theta_2. \tag{20}$$

Step 1: Write the Likelihood Function

Since the observations are independent, the **likelihood function** is:

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\theta_2 - \theta_1}, \quad \text{for } \theta_1 \leq X_i \leq \theta_2. \tag{21}$$

This simplifies to:

$$L(\theta_1, \theta_2) = \frac{1}{(\theta_2 - \theta_1)^n}, \quad \text{where } \theta_1 \leq X_{\min}, \quad \theta_2 \geq X_{\max}. \tag{22}$$

where:

$$X_{\min} = \min(X_1, \dots, X_n), \quad X_{\max} = \max(X_1, \dots, X_n). \tag{23}$$

Step 2: Find the MLE for θ_1 and θ_2

The likelihood function is maximized by **minimizing** the denominator $\theta_2 - \theta_1$, which happens when:

$$\theta_1 = X_{\min}, \quad \theta_2 = X_{\max}. \tag{24}$$

Final Answer

Maximum Likelihood Estimators

$$\hat{\theta}_1 = X_{\min}, \quad \hat{\theta}_2 = X_{\max}. \tag{25}$$

Thus, the **MLE for θ_1 is the smallest observed value**, and the **MLE for θ_2 is the largest observed value**.

Problem 4

Suppose that $X_1, \dots, X_n$ form a random sample from an exponential distribution for which the value of the parameter λ is unknown. Determine the MLE of the median of the distribution.

Solution

We are given a **random sample** $X_1, \dots, X_n$ from an **Exponential distribution**:

$$X_i \sim \text{Exp}(\lambda), \quad \lambda > 0. \tag{26}$$

The probability density function (PDF) is:

$$f(x \mid \lambda) = \lambda e^{-\lambda x}, \quad x > 0. \tag{27}$$

Step 1: Find the MLE of λ

Since the observations are independent, the **likelihood function** is:

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda X_i}. \tag{28}$$

Taking the natural logarithm:

$$\ell(\lambda) = n \ln \lambda - \lambda \sum_{i=1}^{n} X_i. \tag{29}$$

To find the MLE, take the derivative with respect to λ and set it to zero:

$$\frac{d\ell}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i = 0. \tag{30}$$

Solving for λ:

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} X_i}. \tag{31}$$

Step 2: Find the MLE of the Median Using the Invariance Property

The median of an exponential distribution is found by solving:

$$P(X \leq m) = 0.5. \tag{32}$$

Using the cumulative distribution function (CDF):

$$1 - e^{-\lambda m} = 0.5. \tag{33}$$

Solving for m:

$$m = \frac{\ln 2}{\lambda}. \tag{34}$$

Since the MLE satisfies the **invariance property**, meaning that the MLE of a function of a parameter is simply that function applied to the MLE of the parameter, we substitute $\hat{\lambda}$ into the equation:

$$\hat{m} = \frac{\ln 2}{\hat{\lambda}} = \frac{\ln 2}{n} \sum_{i=1}^{n} X_i. \tag{35}$$

Final Answer

Maximum Likelihood Estimator of the Median

$$\hat{m} = \frac{\ln 2}{n} \sum_{i=1}^{n} X_i. \tag{36}$$

Thus, the **MLE for the median** of an exponential distribution is proportional to the sample mean, obtained using the **invariance property** of MLEs.

Problem 5

Suppose that $X_1, \ldots, X_n$ are a random sample from the Normal distribution with unknown mean μ and unknown variance σ^2. Find the MLE of the 0.95 quantile of the distribution (i.e., the value of a such that $P(X < a) = 0.95$).

Solution

We are given a **random sample** $X_1, \ldots, X_n$ from a **Normal distribution**:

$$X_i \sim \mathcal{N}(\mu, \sigma^2), \quad -\infty < \mu < \infty, \quad \sigma^2 > 0. \tag{37}$$

The cumulative distribution function (CDF) of a normal distribution is:

$$P(X < a) = \Phi\left(\frac{a-\mu}{\sigma}\right). \tag{38}$$

Step 1: Express the 0.95 Quantile in Terms of μ and σ

By definition, the 0.95 quantile a satisfies:

$$P(X < a) = 0.95. \tag{39}$$

Using the standard normal quantile function, we write:

$$a = \mu + z_{0.95}\sigma, \tag{40}$$

where $z_{0.95}$ is the 0.95 quantile of the standard normal distribution, which is approximately:

$$z_{0.95} \approx 1.645. \tag{41}$$

Step 2: Find the MLE Using the Invariance Property

From previous results, the MLEs of μ and σ^2 are:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2. \tag{42}$$

Applying the **invariance property** of MLEs, we substitute $\hat{\mu}$ and $\hat{\sigma}$ into the formula for a:

$$\hat{a} = \hat{\mu} + z_{0.95}\hat{\sigma}. \tag{43}$$

Thus, the MLE of the 0.95 quantile is:

$$\hat{a} = \frac{1}{n}\sum_{i=1}^{n} X_i + 1.645\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2}. \tag{44}$$

Final Answer

Maximum Likelihood Estimator of the 0.95 Quantile

$$\hat{a} = \frac{1}{n}\sum_{i=1}^{n} X_i + 1.645\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2}. \tag{45}$$

Thus, the **MLE for the 0.95 quantile** of a normal distribution is a function of the sample mean and sample standard deviation, obtained using the **invariance property** of MLEs.