STAT 131 - H.W. Section 3.8 (Transformations) - Fall 2024

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Introduction

This document provides the solutions for H.W. Section 3.8 (Transformations) for **STAT 131**, instructed by **Dr. Juhee Lee** at the University of California, Santa Cruz. Solutions were prepared by the TA, **Antonio Aguirre**.

Solution to Problem 1

Given X with p.d.f.

$$f_X(x) = \begin{cases} 3x^2 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and $Y = 1 - X^2$, find the p.d.f. of Y.

Step 1: Indicator Form and Support of X

The p.d.f. of X can be expressed with an indicator:

$$f_X(x) = 3x^2 \cdot \mathbf{1}_{(0,1)}(x),$$

with support $0 < x < 1$.

Step 2: CDF of X

For $0 < x < 1$,

$$F_X(x) = \int_0^x 3t^2\, dt = x^3.$$

Thus,

$$F_X(x) = \begin{cases} 0 & x \leq 0, \\ x^3 & 0 < x < 1, \\ 1 & x \geq 1. \end{cases}$$

Step 3: Support of Y

Since $Y = 1 - X^2$ and $0 < X < 1$, we have $Y \in (0, 1)$.

Step 4: CDF of Y in Terms of X

$$F_Y(y) = P(Y \leq y) = P(1 - X^2 \leq y) = P(X^2 \geq 1 - y) = P\left(X \geq \sqrt{1-y}\right).$$

Thus,

$$F_Y(y) = 1 - F_X\left(\sqrt{1-y}\right) = 1 - \left(\sqrt{1-y}\right)^3 = 1 - (1-y)^{3/2}.$$

Step 5: p.d.f. of Y

Differentiating $F_Y(y)$ with respect to y:

$$f_Y(y) = \frac{d}{dy}\left(1 - (1-y)^{3/2}\right) = \frac{3}{2}(1-y)^{1/2}.$$

So, the p.d.f. of Y is:

$$f_Y(y) = \frac{3}{2}(1-y)^{1/2}, \quad 0 < y < 1.$$

Step 6: Verification of Validity

To confirm $f_Y(y)$ integrates to 1:

$$\int_0^1 \frac{3}{2}(1-y)^{1/2}\,dy.$$

Using $u = 1 - y$:

$$= \frac{3}{2}\int_0^1 u^{1/2}\,du = \frac{3}{2}\cdot\frac{2}{3}u^{3/2}\Big|_0^1 = 1.$$

Final Answer

The p.d.f. of $Y = 1 - X^2$ is:

$$f_Y(y) = \frac{3}{2}(1-y)^{1/2}, \quad 0 < y < 1.$$

Solution to Question 2

Suppose that a random variable X can have each of the values $-3, -2, -1, 0, 1, 2, 3$ with equal probability. We are asked to determine the probability mass function (p.m.f.) of $Y = X^2 - X$.

Step 1: Determine Possible Values of Y

Let's compute $Y = X^2 - X$ for each possible value of X:

- If $X = -3$, $Y = (-3)^2 - (-3) = 9 + 3 = 12$
- If $X = -2$, $Y = (-2)^2 - (-2) = 4 + 2 = 6$
- If $X = -1$, $Y = (-1)^2 - (-1) = 1 + 1 = 2$
- If $X = 0$, $Y = 0^2 - 0 = 0$
- If $X = 1$, $Y = 1^2 - 1 = 0$
- If $X = 2$, $Y = 2^2 - 2 = 4 - 2 = 2$
- If $X = 3$, $Y = 3^2 - 3 = 9 - 3 = 6$

Thus, the possible values of Y are $0, 2, 6$, and 12.

Step 2: Determine the Probability of Each Value of Y

Since each value of X occurs with probability $\frac{1}{7}$, we calculate the probability of each unique value of Y by counting how often each occurs.

- **For $Y = 0$:** This occurs when $X = 0$ or $X = 1$.

$$\Pr(Y = 0) = \Pr(X = 0) + \Pr(X = 1) = \frac{1}{7} + \frac{1}{7} = \frac{2}{7}$$

- **For $Y = 2$:** This occurs when $X = -1$ or $X = 2$.

$$\Pr(Y = 2) = \Pr(X = -1) + \Pr(X = 2) = \frac{1}{7} + \frac{1}{7} = \frac{2}{7}$$

- **For $Y = 6$:** This occurs when $X = -2$ or $X = 3$.

$$\Pr(Y = 6) = \Pr(X = -2) + \Pr(X = 3) = \frac{1}{7} + \frac{1}{7} = \frac{2}{7}$$

- **For $Y = 12$:** This occurs only when $X = -3$.

$$\Pr(Y = 12) = \Pr(X = -3) = \frac{1}{7}$$

Step 3: Probability Mass Function of Y

The p.m.f. of Y is:

$$\Pr(Y = y) = \begin{cases} \frac{2}{7}, & y = 0, \\ \frac{2}{7}, & y = 2, \\ \frac{2}{7}, & y = 6, \\ \frac{1}{7}, & y = 12, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 3

Given:

$$f_X(x) = \begin{cases} \frac{1}{2}x & \text{for } 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = X(2-X)$. We aim to find the cumulative distribution function (CDF) and probability density function (PDF) of Y.

Step 1: PDF of X with Indicator

The PDF of X can be written as:

$$f_X(x) = \frac{1}{2}x \cdot \mathbf{1}_{(0,2)}(x),$$

where the support of X is $0 < x < 2$.

Step 2: Transformation and Support of Y

The transformation is $Y = X(2-X) = 2X - X^2$. To determine the range of Y:

- When $X = 0$, $Y = 0$.
- When $X = 1$, $Y = 1$.

Thus, the support of Y is $0 < Y < 1$.

Step 3: CDF of Y in Terms of the CDF of X

To find $F_Y(y) = \Pr(Y \leq y)$, consider two cases for X:

1. **Case $0 < X \leq 1$:** In this range, $Y = X(2-X)$ is increasing. Solving $Y = X(2-X)$ for X gives:

$$X = 1 - \sqrt{1-y}.$$

Thus,

$$F_Y(y) = \Pr(Y \leq y) = \Pr\left(X \leq 1 - \sqrt{1-y}\right) = F_X(1 - \sqrt{1-y}).$$

2. **Case $1 < X < 2$:** In this range, $Y = X(2-X)$ is decreasing. Solving for X:

$$X = 1 + \sqrt{1-y}.$$

So,

$$F_Y(y) = \Pr(Y \leq y) = \Pr\left(X \geq 1 + \sqrt{1-y}\right) = 1 - F_X(1 + \sqrt{1-y}).$$

Combining these cases, we have:

$$F_Y(y) = \left\{ F_X(1 - \sqrt{1-y}) + \left(1 - F_X(1 + \sqrt{1-y})\right), \quad \text{if } 0 < y < 1. \right.$$

Step 4: PDF of Y by Differentiation

Differentiate $F_Y(y)$ with respect to y to obtain $f_Y(y)$:

$$F_Y(y) = 1 - \sqrt{1-y}.$$

Taking the derivative:

$$f_Y(y) = \frac{d}{dy}\left(1 - \sqrt{1-y}\right) = \frac{1}{2\sqrt{1-y}}.$$

Thus, the PDF of Y is:

$$f_Y(y) = \frac{1}{2\sqrt{1-y}}, \quad 0 < y < 1.$$

Step 5: Verify $f_Y(y)$ is a Valid PDF

To confirm that $f_Y(y)$ integrates to 1 over $(0, 1)$:

$$\int_0^1 f_Y(y)\,dy = \int_0^1 \frac{1}{2\sqrt{1-y}}\,dy.$$

Using the substitution $u = 1 - y$, $du = -dy$:

$$= \int_1^0 -\frac{1}{2\sqrt{u}}\,du = \int_0^1 \frac{1}{2\sqrt{u}}\,du.$$

Evaluating this integral:

$$= \frac{1}{2} \cdot 2\sqrt{u}\Big|_0^1 = \sqrt{u}\Big|_0^1 = 1.$$

Thus, $f_Y(y)$ is a valid PDF.

Final Answer

The PDF of $Y = X(2 - X)$ is:

$$f_Y(y) = \frac{1}{2\sqrt{1-y}}, \quad 0 < y < 1.$$

Solution to Problem 8

Given the p.d.f. of X:

$$f_X(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{for } x \le 0, \end{cases}$$

we want to determine the p.d.f. of $Y = X^{1/2}$.

Step 1: Support of Y

Since $X > 0$, it follows that $Y = \sqrt{X} > 0$. Therefore, the support of Y is $y > 0$.

Step 2: Find the CDF of Y

Let $F_Y(y)$ denote the cumulative distribution function of Y. Then,

$$F_Y(y) = \Pr(Y \le y) = \Pr(\sqrt{X} \le y).$$

Squaring both sides within the probability gives:

$$F_Y(y) = \Pr(X \le y^2).$$

Since $F_X(x)$ is the CDF of X, and given that $X \sim \text{Exp}(1)$ (with CDF $F_X(x) = 1 - e^{-x}$ for $x \ge 0$), we have:

$$F_Y(y) = F_X(y^2) = 1 - e^{-y^2} \quad \text{for } y > 0.$$

Step 3: Derive the p.d.f. of Y

To find the p.d.f. $f_Y(y)$, we take the derivative of $F_Y(y)$ with respect to y:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(1 - e^{-y^2}\right).$$

Differentiating, we get:

$$f_Y(y) = e^{-y^2} \cdot 2y = 2ye^{-y^2} \quad \text{for } y > 0.$$

Final Answer

The p.d.f. of $Y = X^{1/2}$ is:

$$f_Y(y) = \begin{cases} 2ye^{-y^2} & \text{for } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Solution to Problem 9

Suppose that X has the uniform distribution on the interval $[0, 1]$. Construct a random variable $Y = r(X)$ for which the p.d.f. of Y will be

$$g(y) = \begin{cases} \frac{3}{8}y^{1/2} & \text{for } 0 < y < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Solution

To solve this, we assume $r(x)$ is invertible and increasing. We start by deriving the c.d.f. of Y in terms of X, and then we find $r(x)$ such that Y has the desired distribution.

Step 1: Cumulative Distribution Function (C.D.F.) of X

Since $X \sim \text{Uniform}(0, 1)$, the p.d.f. and c.d.f. of X are:

$$f_X(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0, \\ x & \text{for } 0 \le x \le 1, \\ 1 & \text{for } x > 1. \end{cases}$$

Step 2: Desired C.D.F. of Y

Since $g(y) = \frac{3}{8}y^{1/2}$ on $0 < y < 2$, we calculate $F_Y(y)$ by integrating $g(y)$:

$$F_Y(y) = \int_0^y \frac{3}{8}t^{1/2}\,dt = \frac{3}{8} \cdot \frac{2}{3}y^{3/2} = \frac{y^{3/2}}{4}.$$

Thus,

$$F_Y(y) = \begin{cases} 0 & \text{for } y \le 0, \\ \frac{y^{3/2}}{4} & \text{for } 0 < y < 2, \\ 1 & \text{for } y \ge 2. \end{cases}$$

Step 3: Find $r(x)$

To match $F_Y(y) = F_X(r^{-1}(y))$, we set $F_Y(y) = x$ and solve for y in terms of x:

$$x = \frac{y^{3/2}}{4}.$$

Solving for y:

$$y = (4x)^{2/3}.$$

Thus, the transformation is

$$Y = r(X) = (4X)^{2/3}.$$

Step 4: Verification of the p.d.f. of Y

To confirm, we find $f_Y(y)$ by using the change of variables formula.

1. Compute $r^{-1}(y)$:

$$X = r^{-1}(y) = \frac{y^{3/2}}{4}.$$

2. Calculate $\frac{d}{dy}r^{-1}(y)$:

$$\frac{d}{dy}r^{-1}(y) = \frac{d}{dy}\left(\frac{y^{3/2}}{4}\right) = \frac{3}{8}y^{1/2}.$$

3. Then,

$$f_Y(y) = f_X(r^{-1}(y))\left|\frac{d}{dy}r^{-1}(y)\right| = 1 \cdot \frac{3}{8}y^{1/2} = \frac{3}{8}y^{1/2}.$$

This matches the desired p.d.f., so our transformation is verified.

Final Answer

The required transformation is

$$Y = r(X) = (4X)^{2/3}.$$

Solution to Question 13

Given the p.d.f. of Z:

$$f_Z(z) = \begin{cases} 2e^{-2z} & \text{for } z > 0, \\ 0 & \text{otherwise.} \end{cases}$$

we want to determine the p.d.f. of $T = \frac{1}{Z}$.

Step 1: CDF of Z

To find the CDF $F_Z(z) = \Pr(Z \leq z)$ for $z > 0$:

$$F_Z(z) = \int_0^z 2e^{-2u}\,du = \left[-e^{-2u}\right]_0^z = 1 - e^{-2z}.$$

Thus, the CDF of Z is:

$$F_Z(z) = \begin{cases} 1 - e^{-2z} & \text{for } z \geq 0, \\ 0 & \text{for } z < 0. \end{cases}$$

Step 2: CDF of $T = \frac{1}{Z}$

We have $F_T(t) = \Pr(T \leq t) = \Pr\left(\frac{1}{Z} \leq t\right)$. Rearranging $\frac{1}{Z} \leq t$ gives $Z \geq \frac{1}{t}$, so:

$$F_T(t) = \Pr\left(Z \geq \frac{1}{t}\right) = 1 - F_Z\left(\frac{1}{t}\right).$$

Substituting $F_Z\left(\frac{1}{t}\right) = 1 - e^{-\frac{2}{t}}$:

$$F_T(t) = e^{-\frac{2}{t}}, \quad \text{for } t > 0.$$

Step 3: PDF of T

Differentiate $F_T(t)$ to find $f_T(t)$:

$$f_T(t) = \frac{d}{dt}F_T(t) = e^{-\frac{2}{t}} \cdot \frac{2}{t^2} = \frac{2}{t^2}e^{-\frac{2}{t}}, \quad \text{for } t > 0.$$

Verification

To confirm $f_T(t)$ is a valid p.d.f., we check:

$$\int_0^\infty \frac{2}{t^2}e^{-\frac{2}{t}}\,dt = 1.$$

Letting $u = \frac{2}{t}$, then $t = \frac{2}{u}$ and $dt = -\frac{2}{u^2}\,du$:

$$\int_0^\infty \frac{2}{t^2}e^{-\frac{2}{t}}\,dt = \int_\infty^0 e^{-u}\,du = 1.$$

Thus, $f_T(t) = \frac{2}{t^2}e^{-\frac{2}{t}}$ is a valid p.d.f. for $t > 0$.