STAT 131 Quiz 3 Solutions - Fall 2024

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Introduction

This document presents solutions for Questions 1 and 2 in all versions of Quiz 3 for STAT 131, instructed by Dr. Juhee Lee at the University of California, Santa Cruz. Antonio Aguirre (TA) prepared the solutions.

Exercise: Linear Combinations of Random Variables

Suppose X and Y are random variables for which $\mathbb{E}(X) = 3$, $\mathbb{E}(Y) = 4$, $\text{Var}(X) = 1$, and $\text{Var}(Y) = 9$.

(a) Assuming Independence

Step 1: Compute $\mathbb{E}(X - 3Y + 5)$

Using the linearity of expectation:

$$\mathbb{E}(X - 3Y + 5) = \mathbb{E}(X) - 3\mathbb{E}(Y) + \mathbb{E}(5).$$

Substituting the given values:

$$\mathbb{E}(X - 3Y + 5) = 3 - 3(4) + 5 = -4.$$

Step 2: Compute Var$(X - 3Y + 5)$

Since X and Y are independent, the variance of a sum is the sum of variances:

$$\text{Var}(X - 3Y + 5) = \text{Var}(X) + \text{Var}(-3Y).$$

Recall that $\text{Var}(aY) = a^2\text{Var}(Y)$, so:

$$\text{Var}(-3Y) = (-3)^2 \cdot \text{Var}(Y) = 9 \cdot 9 = 81.$$

Thus:

$$\text{Var}(X - 3Y + 5) = 1 + 81 = 82.$$

(b) Assuming $\rho(X, Y) = 0.30$

Step 1: Compute $\mathbb{E}(X - 3Y + 5)$

The expectation remains unchanged, as covariance does not affect expectation:

$$\mathbb{E}(X - 3Y + 5) = 3 - 3(4) + 5 = -4.$$

Step 2: Compute Var$(X - 3Y + 5)$

For dependent variables, use the variance formula for a linear combination:

$$\text{Var}(X - 3Y + 5) = \text{Var}(X) + \text{Var}(-3Y) + 2\text{Cov}(X, -3Y).$$

We already know $\text{Var}(-3Y) = 81$. The covariance term is given by:

$$\text{Cov}(X, -3Y) = -3 \cdot \text{Cov}(X, Y).$$

Since $\text{Cov}(X, Y) = \rho(X, Y) \cdot \sqrt{\text{Var}(X) \cdot \text{Var}(Y)}$, substitute the values:

$$\text{Cov}(X, Y) = 0.30 \cdot \sqrt{1 \cdot 9} = 0.30 \cdot 3 = 0.9.$$

$$\text{Cov}(X, -3Y) = -3 \cdot 0.9 = -2.7.$$

Substitute back into the variance formula:

$$\text{Var}(X - 3Y + 5) = 1 + 81 + 2(-2.7) = 1 + 81 - 5.4 = 76.6.$$

Final Results

- **(a) Independence:**

$$\mathbb{E}(X - 3Y + 5) = -4, \quad \text{Var}(X - 3Y + 5) = 82.$$

- **(b)** $\rho(X, Y) = 0.30$**:**

$$\mathbb{E}(X - 3Y + 5) = -4, \quad \text{Var}(X - 3Y + 5) = 76.6.$$

Question 1B

Suppose X and Y are random variables for which $\mathbb{E}(X) = 4$, $\mathbb{E}(Y) = 3$, $\text{Var}(X) = 4$, and $\text{Var}(Y) = 9$.

(a) Assuming Independence

Step 1: Compute $\mathbb{E}(X - 2Y - 3)$

Using the linearity of expectation:

$$\mathbb{E}(X - 2Y - 3) = \mathbb{E}(X) - 2\mathbb{E}(Y) - 3.$$

Substituting the given values:

$$\mathbb{E}(X - 2Y - 3) = 4 - 2(3) - 3 = -5.$$

Step 2: Compute Var$(X - 2Y - 3)$

Since X and Y are independent, the variance of a sum is the sum of variances:

$$\text{Var}(X - 2Y - 3) = \text{Var}(X) + \text{Var}(-2Y).$$

Using $\text{Var}(aY) = a^2\text{Var}(Y)$:

$$\text{Var}(-2Y) = (-2)^2 \cdot \text{Var}(Y) = 4 \cdot 9 = 36.$$

Thus:

$$\text{Var}(X - 2Y - 3) = 4 + 36 = 40.$$

(b) Assuming $\rho(X, Y) = -0.30$

Step 1: Compute $\mathbb{E}(X - 2Y - 3)$

The expectation remains unchanged:

$$\mathbb{E}(X - 2Y - 3) = 4 - 2(3) - 3 = -5.$$

Step 2: Compute Var$(X - 2Y - 3)$

For dependent variables, use the variance formula:

$$\text{Var}(X - 2Y - 3) = \text{Var}(X) + \text{Var}(-2Y) + 2\text{Cov}(X, -2Y).$$

From part (a), $\text{Var}(-2Y) = 36$. For $\text{Cov}(X, -2Y)$:

$$\text{Cov}(X, -2Y) = -2 \cdot \text{Cov}(X, Y),$$

and:

$$\text{Cov}(X, Y) = \rho(X, Y) \cdot \sqrt{\text{Var}(X) \cdot \text{Var}(Y)}.$$

Substitute the values:

$$\mathrm{Cov}(X, Y) = -0.30 \cdot \sqrt{4 \cdot 9} = -0.30 \cdot 6 = -1.8,$$

$$\mathrm{Cov}(X, -2Y) = -2 \cdot (-1.8) = 3.6.$$

Thus:

$$\mathrm{Var}(X - 2Y - 3) = 4 + 36 + 2(3.6) = 4 + 36 + 7.2 = 47.2.$$

Final Results

- **(a) Independence:**

$$\mathbb{E}(X - 2Y - 3) = -5, \quad \mathrm{Var}(X - 2Y - 3) = 40.$$

- **(b)** $\rho(X, Y) = -0.30$**:**

$$\mathbb{E}(X - 2Y - 3) = -5, \quad \mathrm{Var}(X - 2Y - 3) = 47.2.$$

Question 1C

Suppose X and Y are random variables for which $\mathbb{E}(X) = 3$, $\mathbb{E}(Y) = 4$, $\text{Var}(X) = 1$, and $\text{Var}(Y) = 9$.

(a) Assuming Independence

Step 1: Compute $\mathbb{E}(2X - Y + 2)$

Using the linearity of expectation:

$$\mathbb{E}(2X - Y + 2) = 2\mathbb{E}(X) - \mathbb{E}(Y) + \mathbb{E}(2).$$

Substituting the given values:

$$\mathbb{E}(2X - Y + 2) = 2(3) - 4 + 2 = 4.$$

Step 2: Compute Var$(2X - Y + 2)$

Since X and Y are independent, the variance of a sum is the sum of variances:

$$\text{Var}(2X - Y + 2) = \text{Var}(2X) + \text{Var}(-Y).$$

Using $\text{Var}(aX) = a^2\text{Var}(X)$:

$$\text{Var}(2X) = 2^2 \cdot \text{Var}(X) = 4 \cdot 1 = 4, \quad \text{Var}(-Y) = (-1)^2 \cdot \text{Var}(Y) = 1 \cdot 9 = 9.$$

Thus:

$$\text{Var}(2X - Y + 2) = 4 + 9 = 13.$$

(b) Assuming $\rho(X, Y) = 0.40$

Step 1: Compute $\mathbb{E}(2X - Y + 2)$

The expectation remains unchanged:

$$\mathbb{E}(2X - Y + 2) = 2\mathbb{E}(X) - \mathbb{E}(Y) + 2 = 4.$$

Step 2: Compute Var$(2X - Y + 2)$

For dependent variables, use the variance formula:

$$\text{Var}(2X - Y + 2) = \text{Var}(2X) + \text{Var}(-Y) + 2\text{Cov}(2X, -Y).$$

From part (a), $\text{Var}(2X) = 4$ and $\text{Var}(-Y) = 9$. For $\text{Cov}(2X, -Y)$:

$$\text{Cov}(2X, -Y) = 2 \cdot (-1) \cdot \text{Cov}(X, Y),$$

and:

$$\text{Cov}(X, Y) = \rho(X, Y) \cdot \sqrt{\text{Var}(X) \cdot \text{Var}(Y)}.$$

Substitute the values:

$$\text{Cov}(X, Y) = 0.40 \cdot \sqrt{1 \cdot 9} = 0.40 \cdot 3 = 1.2,$$

$$\text{Cov}(2X, -Y) = 2 \cdot (-1) \cdot 1.2 = -2.4.$$

Thus:

$$\text{Var}(2X - Y + 2) = 4 + 9 + 2(-2.4) = 4 + 9 - 4.8 = 8.2.$$

Final Results

- **(a) Independence:**

$$\mathbb{E}(2X - Y + 2) = 4, \quad \text{Var}(2X - Y + 2) = 13.$$

- **(b)** $\rho(X, Y) = 0.40$**:**

$$\mathbb{E}(2X - Y + 2) = 4, \quad \text{Var}(2X - Y + 2) = 8.2.$$

Question 2A

Suppose X and Y are independent, standard normal random variables ($X, Y \sim \mathcal{N}(0,1)$). We aim to evaluate:

$$\Pr(-2.0 < 2X + Y < 4.5).$$

Step 1: Define the Random Variable

Define $Z = 2X + Y$. Since X and Y are independent:

- $\mathbb{E}[Z] = 2\mathbb{E}[X] + \mathbb{E}[Y] = 2(0) + 0 = 0,$
- $\text{Var}(Z) = 2^2\text{Var}(X) + \text{Var}(Y) = 4(1) + 1 = 5.$

Thus, $Z \sim \mathcal{N}(0, 5)$, and $\sigma_Z = \sqrt{5} \approx 2.236$.

Step 2: Standardize the Inequality

We standardize Z to a standard normal variable $Z_s \sim \mathcal{N}(0,1)$:

$$Z_s = \frac{Z - \mu_Z}{\sigma_Z}.$$

Transform the bounds:

$$\begin{aligned} Z_s \text{ for } Z = -2.0: &\quad Z_s = \frac{-2.0}{2.236} \approx -0.895, \\ Z_s \text{ for } Z = 4.5: &\quad Z_s = \frac{4.5}{2.236} \approx 2.012. \end{aligned}$$

The inequality becomes:

$$\Pr(-2.0 < 2X + Y < 4.5) = \Pr(-0.895 < Z_s < 2.012).$$

Step 3: Use the Standard Normal Table

Using the standard normal cumulative distribution function $\Phi(z)$:

- $\Pr(Z_s \leq 2.012) = \Phi(2.01) \approx 0.9788,$

- $\Pr(Z_s \leq -0.895) = \Phi(-0.89) = 1 - \Phi(0.89) \approx 1 - 0.8133 = 0.1867$. Thus:

$$\Pr(-0.895 < Z_s < 2.012) = \Phi(2.012) - \Phi(-0.895).$$

Substituting:

$$\Pr(-0.895 < Z_s < 2.012) = 0.9788 - 0.1867 = 0.7921.$$

Final Answer

The probability that $-2.0 < 2X + Y < 4.5$ is:

$$\boxed{0.7921}.$$

Question 2B

Suppose X and Y are independent, standard normal random variables ($X, Y \sim \mathcal{N}(0, 1)$). We aim to evaluate:

$$\Pr(-2.5 < X - 2Y < 1).$$

Step 1: Define the Random Variable

Define $Z = X - 2Y$. Since X and Y are independent:

- $\mathbb{E}[Z] = \mathbb{E}[X] - 2\mathbb{E}[Y] = 0 - 2(0) = 0$,
- $\text{Var}(Z) = \text{Var}(X) + (-2)^2\text{Var}(Y) = 1 + 4(1) = 5$.

Thus, $Z \sim \mathcal{N}(0, 5)$, and $\sigma_Z = \sqrt{5} \approx 2.236$.

Step 2: Standardize the Inequality

We standardize Z to a standard normal variable $Z_s \sim \mathcal{N}(0, 1)$:

$$Z_s = \frac{Z - \mu_Z}{\sigma_Z}.$$

Transform the bounds:

$$Z_s \text{ for } Z = -2.5 : \quad Z_s = \frac{-2.5}{2.236} \approx -1.118,$$
$$Z_s \text{ for } Z = 1 : \quad Z_s = \frac{1}{2.236} \approx 0.447.$$

The inequality becomes:

$$\Pr(-2.5 < X - 2Y < 1) = \Pr(-1.118 < Z_s < 0.447).$$

Step 3: Use the Standard Normal Table

Using the standard normal cumulative distribution function $\Phi(z)$:

- $\Pr(Z_s \leq 0.447) = \Phi(0.45) \approx 0.6736$,
- $\Pr(Z_s \leq -1.118) = \Phi(-1.12) = 1 - \Phi(1.12) \approx 1 - 0.8686 = 0.1314$. Thus:

$$\Pr(-1.118 < Z_s < 0.447) = \Phi(0.447) - \Phi(-1.118).$$

Substituting:

$$\Pr(-1.118 < Z_s < 0.447) = 0.6736 - 0.1314 = 0.5422.$$

Final Answer

The probability that $-2.5 < X - 2Y < 1$ is:

$$\boxed{0.5422}.$$

Question 2C

Suppose X and Y are independent standard normal random variables ($X, Y \sim \mathcal{N}(0,1)$). Evaluate $\Pr(-1.0 < 2X - Y < 2.0)$.

Step 1: Distribution of $Z = 2X - Y$

We define $Z = 2X - Y$. Using the properties of expectation and variance:

- $\mathbb{E}[Z] = \mathbb{E}[2X - Y] = 2\mathbb{E}[X] - \mathbb{E}[Y] = 0,$
- $\text{Var}(Z) = \text{Var}(2X) + \text{Var}(-Y) = 4 + 1 = 5.$

Thus, $Z \sim \mathcal{N}(0, 5)$, and the standard deviation is $\sigma_Z = \sqrt{5} \approx 2.236$.

Step 2: Standardize the Inequality

To compute $\Pr(-1.0 < Z < 2.0)$, we standardize Z to a standard normal variable $Z_s \sim \mathcal{N}(0,1)$ using:

$$Z_s = \frac{Z - \mu_Z}{\sigma_Z}.$$

Transform the bounds:

$$Z_s = \frac{-1.0}{\sqrt{5}} \approx -0.447, \quad Z_s = \frac{2.0}{\sqrt{5}} \approx 0.894.$$

Thus:

$$\Pr(-1.0 < Z < 2.0) = \Pr(-0.447 < Z_s < 0.894).$$

Step 3: Use the Standard Normal Table

Using the standard normal table:

- $\Phi(0.894)$: For $z = 0.89$, $\Phi(0.89) \approx 0.8133$,
- $\Phi(-0.447)$: Using symmetry, $\Phi(-z) = 1 - \Phi(z)$. Look up $\Phi(0.45)$, which gives $\Phi(0.45) \approx 0.6736$. Thus, $\Phi(-0.447) = 1 - 0.6736 = 0.3264$.

The probability is:

$$\Pr(-0.447 < Z_s < 0.894) = \Phi(0.894) - \Phi(-0.447).$$

Substituting the values:

$$\Pr(-0.447 < Z_s < 0.894) = 0.8133 - 0.3264 = 0.4869.$$

Final Answer

The probability that $-1.0 < 2X - Y < 2.0$ is:

$$\boxed{0.4869}.$$