STAT 131 Quiz 1 Solutions - Fall 2024

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Question 1 Part (a) - Version A

Problem Statement: Suppose that a joint p.d.f. of two random variables X and Y is as follows:

$$f(x,y) = \begin{cases} \frac{3}{8}(x^2+y) & \text{for } 0<x<1 \text{ and } 0<y<2, \\ 0 & \text{otherwise.} \end{cases}$$

Determine $\Pr(X < \frac{1}{4}|Y=1)$.
Solution:

1. **Determine the marginal distribution of Y:**

$$\begin{aligned} f_Y(y) &= \int_0^1 f(x,y)\,dx = \int_0^1 \frac{3}{8}(x^2+y)\,dx \\ &= \frac{3}{8}\int_0^1 (x^2+y)\,dx = \frac{3}{8}\left(\int_0^1 x^2\,dx + \int_0^1 y\,dx\right) \\ &= \frac{3}{8}\left(\left.\frac{x^3}{3}\right|_0^1 + y\,x|_0^1\right) = \frac{3}{8}\left(\frac{1}{3} + y\cdot 1\right) \\ &= \frac{3}{8}\left(y+\frac{1}{3}\right) = \frac{3}{8}y + \frac{1}{8} \end{aligned}$$

2. **Find the conditional distribution of X given Y:**

$$f_{X|Y}(x\,|\,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{3}{8}(x^2+y)}{\frac{3}{8}y+\frac{1}{8}} = \frac{3(x^2+y)}{3y+1}$$

3. **Plug in the value $Y=1$:**

$$f_{X|Y}(x\,|\,y=1) = \frac{3}{4}(x^2+1)$$

4. **Find the conditional CDF X given $Y = 1$:**

$$
\begin{aligned}
F_{X|Y}(x \mid Y=1) &= \int_0^x f_{X|Y}(t \mid Y=1)\,dt \;=\; \int_0^x \frac{3}{4}(t^2+1)\,dt \\
&= \frac{3}{4}\int_0^x (t^2+1)\,dt \;=\; \frac{3}{4}\left[\frac{t^3}{3}+t\right]_0^x \\
&= \frac{3}{4}\left(\frac{x^3}{3}+x-0\right) \;=\; \frac{3}{4}\left(\frac{x^3}{3}+x\right) \\
&= \frac{3}{4}x+\frac{x^3}{4}
\end{aligned}
$$

5. **Calculate the final probability:**

$$
\begin{aligned}
\Pr\left(X < \tfrac{1}{4} \mid Y=1\right) = F_{X|Y}\left(\tfrac{1}{4} \mid Y=1\right) \;&=\; \frac{3}{4}\cdot\frac{1}{4}+\frac{\left(\frac{1}{4}\right)^3}{4} \\
&= \frac{3}{16}+\frac{1}{256} \;=\; \frac{48}{256}+\frac{1}{256} \\
&= \frac{49}{256}
\end{aligned}
$$

Question 1 Part (b) - Version A

Problem Statement: Are the random variables X and Y independent? Explain why or why not.
Solution:

1. **Compute the marginal distribution of X:**

$$
\begin{aligned}
f_X(x) &= \int_0^2 f(x,y)\,dy = \int_0^2 \frac{3}{8}(x^2+y)\,dy \\
&= \frac{3}{8}\int_0^2 (x^2+y)\,dy = \frac{3}{8}\left(x^2\int_0^2 dy + \int_0^2 y\,dy\right) \\
&= \frac{3}{8}\left(x^2\cdot 2 + \left.\frac{y^2}{2}\right|_0^2\right) = \frac{3}{8}\left(2x^2+\frac{4}{2}\right) \\
&= \frac{3}{8}\left(2x^2+2\right) = \frac{3}{4}(x^2+1)
\end{aligned}
$$

2. **Compare $f(x, y)$ and $f_X(x)f_Y(y)$:**

   The form of $f(x, y)$ cannot be factored into two functions $h_X(x)h_Y(y)$ because of the addition within $\frac{3}{8}(x^2 + y)$. Furthermore, $f(x, y)$ is not equal to $f_X(x)f_Y(y)$. That is, $\frac{3}{8}(x^2 + y) \neq \frac{3}{4}(x^2 + 1) \times \frac{1}{8}(3y + 1)$. Hence, we conclude that X and Y are **not independent**.

Question 1 Part (a) - Version B

Problem Statement: Suppose that a joint p.d.f. of two random variables X and Y is as follows:

$$f(x,y) = \begin{cases} \frac{3}{11}(x+y^2) & \text{for } 0 < x < 1 \text{ and } 0 < y < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Determine $\Pr\left(X < \frac{1}{4} \mid Y = 1\right)$.

Solution:

1. **Determine the marginal distribution of Y:**

$$\begin{aligned} f_Y(y) &= \int_0^1 f(x,y)\,dx = \int_0^1 \frac{3}{11}(x+y^2)\,dx \\ &= \frac{3}{11}\int_0^1 (x+y^2)\,dx = \frac{3}{11}\left(\int_0^1 x\,dx + \int_0^1 y^2\,dx\right) \\ &= \frac{3}{11}\left(\left.\frac{x^2}{2}\right|_0^1 + y^2\, x|_0^1\right) = \frac{3}{11}\left(\frac{1}{2} + y^2\cdot 1\right) \\ &= \frac{3}{11}\left(y^2 + \frac{1}{2}\right) \end{aligned}$$

2. **Find the conditional distribution of X given Y:**

$$\begin{aligned} f_{X|Y}(x \mid y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{3}{11}(x+y^2)}{\frac{3}{11}\left(y^2+\frac{1}{2}\right)} \\ &= \frac{x+y^2}{y^2+\frac{1}{2}} \end{aligned}$$

3. **Plug in the value $Y = 1$:**

$$f_{X|Y}(x \mid y=1) = \frac{x+1}{1+\frac{1}{2}} = \frac{2}{3}(x+1)$$

4. **Find the conditional CDF of X given $Y = 1$:**

$$\begin{aligned} F_{X|Y}(x \mid Y=1) &= \int_0^x f_{X|Y}(t \mid Y=1)\,dt = \int_0^x \frac{2}{3}(t+1)\,dt \\ &= \frac{2}{3}\int_0^x (t+1)\,dt = \frac{2}{3}\left[\frac{t^2}{2}+t\right]_0^x \\ &= \frac{2}{3}\left(\frac{x^2}{2}+x-0\right) = \frac{2}{3}\left(\frac{x^2}{2}+x\right) \\ &= \frac{x^2}{3} + \frac{2x}{3} \end{aligned}$$

5. **Calculate the final probability:**

$$\begin{aligned}
\Pr\left(X < \frac{1}{4} \mid Y = 1\right) = F_{X|Y}\left(\frac{1}{4} \mid Y = 1\right) &= \frac{\left(\frac{1}{4}\right)^2}{3} + \frac{2\left(\frac{1}{4}\right)}{3} \\
&= \frac{\frac{1}{16}}{3} + \frac{\frac{1}{2}}{3} = \frac{1}{48} + \frac{1}{6} \\
&= \frac{1}{48} + \frac{8}{48} = \frac{9}{48} \\
&= \frac{3}{16}
\end{aligned}$$

Question 1 Part (b) - Version B

Problem Statement: Are the random variables X and Y independent? Explain why or why not.
Solution:

1. **Compute the marginal distribution of X:**

$$\begin{aligned}
f_X(x) &= \int_0^2 f(x,y)\,dy = \int_0^2 \frac{3}{11}(x + y^2)\,dy \\
&= \frac{3}{11}\int_0^2 (x + y^2)\,dy = \frac{3}{11}\left(x \cdot 2 + \frac{y^3}{3}\bigg|_0^2\right) \\
&= \frac{3}{11}\left(2x + \frac{8}{3}\right)
\end{aligned}$$

2. **Compare $f(x, y)$ and $f_X(x)f_Y(y)$:**

   The form of $f(x, y)$ cannot be factored into two functions $h_X(x)h_Y(y)$ because of the addition within $\frac{3}{11}(x + y^2)$. Furthermore, $f(x, y)$ is not equal to $f_X(x)f_Y(y)$. That is, $\frac{3}{11}(x + y^2) \neq \frac{3}{11}(2x + 8/3) \times \frac{3}{11}(y^2 + 1/2)$. Hence, we conclude that X and Y are **not independent**.

Question 1 Part (a) - Version C

Problem Statement: Suppose that a joint p.d.f. of two random variables X and Y is as follows:

$$f(x,y) = \begin{cases} \frac{1}{10}(3x^2+2y) & \text{for } 0 < x < 2 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine $\Pr\left(X < 1 \mid Y = \frac{1}{4}\right)$.

Solution:

1. **Determine the marginal distribution of Y:**

$$\begin{aligned} f_Y(y) &= \int_0^2 f(x,y)\,dx = \int_0^2 \frac{1}{10}(3x^2+2y)\,dx \\ &= \frac{1}{10}\int_0^2 (3x^2+2y)\,dx = \frac{1}{10}\left(\int_0^2 3x^2\,dx + \int_0^2 2y\,dx\right) \\ &= \frac{1}{10}\left(x^3\big|_0^2 + 2y\,x\big|_0^2\right) = \frac{1}{10}\left(8+2y\cdot 2\right) \\ &= \frac{1}{10}\left(8+4y\right) = \frac{8+4y}{10} \end{aligned}$$

2. **Find the conditional distribution of X given Y:**

$$\begin{aligned} f_{X|Y}(x \mid y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{10}(3x^2+2y)}{\frac{8+4y}{10}} \\ &= \frac{3x^2+2y}{8+4y} \end{aligned}$$

3. **Plug in the value $Y = \frac{1}{4}$:**

$$f_{X|Y}\left(x \mid Y = \frac{1}{4}\right) = \frac{3x^2 + 2\left(\frac{1}{4}\right)}{8+4\left(\frac{1}{4}\right)} = \frac{3x^2+\frac{1}{2}}{8+1} = \frac{3x^2+\frac{1}{2}}{9}$$

4. **Find the conditional CDF of X given $Y = \frac{1}{4}$:**

$$\begin{aligned} F_{X|Y}\left(x \mid Y = \frac{1}{4}\right) &= \int_0^x f_{X|Y}\left(t \mid Y = \frac{1}{4}\right)\,dt = \int_0^x \frac{3t^2+\frac{1}{2}}{9}\,dt \\ &= \frac{1}{9}\int_0^x \left(3t^2+\frac{1}{2}\right)dt = \frac{1}{9}\left[t^3+\frac{1}{2}t\right]_0^x \\ &= \frac{1}{9}\left(x^3+\frac{1}{2}x-0\right) = \frac{x^3}{9}+\frac{x}{18} \end{aligned}$$

5. **Calculate the final probability:**

$$\Pr\left(X < 1 \mid Y = \frac{1}{4}\right) = F_{X|Y}\left(1 \mid Y = \frac{1}{4}\right) = \frac{1^3}{9} + \frac{1}{18}$$
$$= \frac{1}{9} + \frac{1}{18} = \frac{2}{18} + \frac{1}{18} = \frac{3}{18}$$
$$= \frac{1}{6}$$

Question 1 Part (b) - Version C

Problem Statement: Are the random variables X and Y independent? Explain why or why not.
Solution:

1. **Compute the marginal distribution of X:**

$$f_X(x) = \int_0^1 f(x, y)\, dy = \int_0^1 \frac{1}{10}(3x^2 + 2y)\, dy$$
$$= \frac{1}{10}\int_0^1 (3x^2 + 2y)\, dy = \frac{1}{10}\left(3x^2 \cdot 1 + y^2\Big|_0^1\right)$$
$$= \frac{1}{10}\left(3x^2 + 1\right)$$

2. **Compare $f(x, y)$ and $f_X(x)f_Y(y)$:**

   The form of $f(x, y)$ cannot be factored into two functions $h_X(x)h_Y(y)$ because of the addition within $\frac{1}{10}(3x^2 + 2y)$. Furthermore, $f(x, y)$ is not equal to $f_X(x)f_Y(y)$. That is, $\frac{1}{10}(3x^2 + 2y) \neq \frac{1}{10}(3x^2 + 1) \times \frac{1}{10}(8 + 4y)$. Hence, we conclude that X and Y are **not independent**.