# STAT 131 Quiz 1 Solutions - Fall 2024

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November 4, 2024

**University of California, Santa Cruz**
Fall 2024

## Introduction

This document presents solutions for Questions 1, 2, and 3 in all versions of Quiz 1 for STAT 131, instructed by Dr. Juhee Lee at the University of California, Santa Cruz. Antonio Aguirre (TA) prepared the solutions.

## Question 1A

**Problem Statement:** A box contains 18 light bulbs, of which five are defective. If a person selects six bulbs at random, without replacement, what is the probability that one or two bulbs will be defective?

**Solution:**

1. **Calculate the Total Number of Ways to Select 6 Bulbs from 18:**

$$\binom{18}{6} = \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 18564$$

2. **Calculate the Number of Ways to Select 1 Defective and 5 Non-defective Bulbs:**
   The number of ways to choose 1 defective bulb is:

$$\binom{5}{1} = 5$$

   The number of ways to choose 5 non-defective bulbs is:

$$\binom{13}{5} = 1287$$

   Total number of ways:

$$5 \times 1287 = 6435$$

3. **Calculate the Number of Ways to Select 2 Defective and 4 Non-defective Bulbs:**
   The number of ways to choose 2 defective bulbs is:
   $$\binom{5}{2} = 10$$
   The number of ways to choose 4 non-defective bulbs is:
   $$\binom{13}{4} = 715$$
   Total number of ways:
   $$10 \times 715 = 7150$$

4. **Total Number of Ways to Get 1 or 2 Defective Bulbs:**
   $$6435 + 7150 = 13585$$

5. **Probability Calculation:**
   The total number of ways to select any 6 bulbs is $\binom{18}{6} = 18564$, so the probability is:
   $$\text{Probability} = \frac{13585}{18564} \approx 0.7317$$

# Question 1B

**Problem Statement:** A box contains 15 light bulbs, of which four are defective. If a person selects eight bulbs at random, without replacement, what is the probability that one or two bulbs will be defective?

**Solution:**

1. **Calculate the Total Number of Ways to Select 8 Bulbs from 15:**
   $$\binom{15}{8} = \frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 6435$$

2. **Calculate the Number of Ways to Select 1 Defective and 7 Non-defective Bulbs:**
   The number of ways to choose 1 defective bulb is:
   $$\binom{4}{1} = 4$$
   The number of ways to choose 7 non-defective bulbs is:
   $$\binom{11}{7} = 330$$
   Total number of ways:
   $$4 \times 330 = 1320$$

3. **Calculate the Number of Ways to Select 2 Defective and 6 Non-defective Bulbs:**
   The number of ways to choose 2 defective bulbs is:
$$\binom{4}{2} = 6$$
   The number of ways to choose 6 non-defective bulbs is:
$$\binom{11}{6} = 462$$
   Total number of ways:
$$6 \times 462 = 2772$$

4. **Total Number of Ways to Get 1 or 2 Defective Bulbs:**
$$1320 + 2772 = 4092$$

5. **Probability Calculation:**
   The total number of ways to select any 8 bulbs is $\binom{15}{8} = 6435$, so the probability is:
$$\text{Probability} = \frac{4092}{6435} \approx 0.6359$$

# Question 1C

**Problem Statement:** A box contains 13 light bulbs, of which four are defective. If a person selects five bulbs at random, without replacement, what is the probability that one or two bulbs will be defective?

**Solution:**

1. **Calculate the Total Number of Ways to Select 5 Bulbs from 13:**
$$\binom{13}{5} = \frac{13 \times 12 \times 11 \times 10 \times 9}{5 \times 4 \times 3 \times 2 \times 1} = 1287$$

2. **Calculate the Number of Ways to Select 1 Defective and 4 Non-defective Bulbs:**
   The number of ways to choose 1 defective bulb is:
$$\binom{4}{1} = 4$$
   The number of ways to choose 4 non-defective bulbs is:
$$\binom{9}{4} = 126$$
   Total number of ways:
$$4 \times 126 = 504$$

3. **Calculate the Number of Ways to Select 2 Defective and 3 Non-defective Bulbs:**
   The number of ways to choose 2 defective bulbs is:
   $$\binom{4}{2} = 6$$
   The number of ways to choose 3 non-defective bulbs is:
   $$\binom{9}{3} = 84$$
   Total number of ways:
   $$6 \times 84 = 504$$

4. **Total Number of Ways to Get 1 or 2 Defective Bulbs:**
   $$504 + 504 = 1008$$

5. **Probability Calculation:**
   The total number of ways to select any 5 bulbs is $\binom{13}{5} = 1287$, so the probability is:
   $$\text{Probability} = \frac{1008}{1287} \approx 0.7833$$

# Question 2A

**Problem Statement:** Specify whether the following statement is true or false. Justify your answer in a few sentences. No credit will be given for answers without justification.

Consider two events $A$ and $B$ with $A \subset B$. Given $Pr(A) = 1/2$ and $Pr(B) = 2/3$, we want to determine if the statement $Pr(B \cap A^c) = 1/4$ is true or false.

**Solution:**

1. **Decomposing** $B \cap A^c$:
   Since $A \subset B$, we have:
   $$B = A \cup (B \cap A^c)$$
   Therefore, since they are a union of disjoint events:
   $$Pr(B) = Pr(A) + Pr(B \cap A^c)$$

2. **Calculating** $Pr(B \cap A^c)$:
   Plugging in the values $Pr(A) = 1/2$ and $Pr(B) = 2/3$:
   $$Pr(B \cap A^c) = 2/3 - 1/2 = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$$

3. **Conclusion:**
   The statement $Pr(B \cap A^c) = 1/4$ is false, because $Pr(B \cap A^c) = 1/6$.

# Question 2B

**Problem Statement:** Specify whether the following statement is true or false. Justify your answer in a few sentences. No credit will be given for answers without justification.

Consider two events $A$ and $B$ with $Pr(A) = 0.6$ and $Pr(B) = 0.3$. We want to determine if the statement, "The maximum possible value of $Pr(A \cap B) = 0.6$" is true or false.

**Solution:**

1. **Understanding the Maximum Value of** $Pr(A \cap B)$:
   The maximum possible value of $Pr(A \cap B)$ is $\min(Pr(A), Pr(B))$, since their intersection is a subset of each. That is, $A \cap B \subseteq A$ and $A \cap B \subseteq B$, and therefore $Pr(A \cap B) \leq Pr(A)$ and $Pr(A \cap B) \leq Pr(B)$.

2. **Calculating the Maximum Value**:
   Given $Pr(A) = 0.6$ and $Pr(B) = 0.3$, we have:
   $$\min(Pr(A), Pr(B)) = \min(0.6, 0.3) = 0.3$$

3. **Conclusion:**
   The maximum possible value of $Pr(A \cap B)$ is 0.3, not 0.6. The statement is false.

# Question 2C

**Problem Statement:** Specify whether the following statement is true or false. Justify your answer in a few sentences. No credit will be given for answers without justification.

Consider two events $A$ and $B$ with $Pr(A) = 0.6$ and $Pr(B) = 0.5$. We want to determine if the statement, "The minimum possible value of $Pr(A \cap B) = 0.5$" is true or false.

**Solution:**

1. **Understanding the Minimum Value of** $Pr(A \cap B)$:
   The minimum possible value of $Pr(A \cap B)$ is given by:
   $$Pr(A) + Pr(B) - 1$$
   Since $Pr(A \cap B) = Pr(A) + Pr(B) - Pr(A \cup B)$ and $Pr(A \cup B) \leq 1$

2. **Calculating the Minimum Value**:
   Given $Pr(A) = 0.6$ and $Pr(B) = 0.5$, we have:
   $$Pr(A) + Pr(B) - 1 = 0.6 + 0.5 - 1 = 0.1$$

3. **Conclusion:**
   The minimum possible value of $Pr(A \cap B)$ is 0.1, not 0.5. The statement is false.

# Question 3A

**Problem Statement:** Suppose that a box contains five blue cards and five red cards. Suppose also that two of these 10 cards are selected at random, without replacement. If it is known that at least one red card has been selected, what is the probability that both selected cards are red?

**Solution:**

1. **Define Events:**
   Let $R$ be the event that both selected cards are red.
   Let $A$ be the event that at least one red card is selected.

2. **Calculate $Pr(R \cap A)$:**
   The event $R \cap A$ is the same as the event $R$ because if both selected cards are red, then at least one red card is selected.
   The probability of selecting both red cards from the 10 cards is:

   $$Pr(R) = \frac{\binom{5}{2}\binom{5}{0}}{\binom{10}{2}} = \frac{10 \times 1}{45} = \frac{10}{45} = \frac{2}{9}$$

3. **Calculate $Pr(A)$:**
   Probability of selecting at least one red card:
   First, calculate the probability of selecting no red cards (i.e., both cards are blue):

   $$Pr(A^c) = \frac{\binom{5}{2}\binom{5}{0}}{\binom{10}{2}} = \frac{10}{45} = \frac{2}{9}$$

   Thus, the probability of selecting at least one red card is:

   $$Pr(A) = 1 - Pr(A^c) = 1 - \frac{2}{9} = \frac{7}{9}$$

4. **Calculate $Pr(R \mid A)$:**
   By the definition of conditional probability:

   $$Pr(R \mid A) = \frac{Pr(R \cap A)}{Pr(A)} = \frac{Pr(R)}{Pr(A)} = \frac{\frac{2}{9}}{\frac{7}{9}} = \frac{2}{7}$$

5. **Conclusion:**
   The probability that both selected cards are red, given that at least one red card has been selected, is:

   $$\frac{2}{7} \approx 0.2857$$

# Question 3B

**Problem Statement:** Suppose that a box contains five blue cards and four red cards. Suppose also that two of these nine cards are selected at random, without replacement. If it is known that at least one blue card has been selected, what is the probability that both selected cards are blue?

**Solution:**

1. **Define Events:**
   Let $B$ be the event that both selected cards are blue.
   Let $A$ be the event that at least one blue card is selected.

2. **Calculate** $Pr(B \cap A)$:
   The event $B \cap A$ is the same as the event $B$ because if both selected cards are blue, then at least one blue card has been selected.
   The probability of selecting both blue cards from the 9 cards is:
   $$Pr(B) = \frac{\binom{5}{2}\binom{4}{0}}{\binom{9}{2}} = \frac{10 \times 1}{36} = \frac{10}{36} = \frac{5}{18}$$

3. **Calculate** $Pr(A)$:
   Probability of selecting at least one blue card:
   Calculate the probability of not selecting any blue cards (i.e., selecting two red cards):
   $$Pr(A^c) = \frac{\binom{5}{0}\binom{4}{2}}{\binom{9}{2}} = \frac{1 \times 6}{36} = \frac{6}{36} = \frac{1}{6}$$
   Thus, the probability of selecting at least one blue card is:
   $$Pr(A) = 1 - Pr(A^c) = 1 - \frac{1}{6} = \frac{5}{6}$$

4. **Calculate** $Pr(B \mid A)$:
   By the definition of conditional probability:
   $$Pr(B \mid A) = \frac{Pr(B \cap A)}{Pr(A)} = \frac{Pr(B)}{Pr(A)} = \frac{\frac{5}{18}}{\frac{5}{6}} = \frac{5}{18} \times \frac{6}{5} = \frac{1}{3}$$

5. **Conclusion:**
   The probability that both selected cards are blue, given that at least one blue card has been selected, is:
   $$\frac{1}{3} \approx 0.3333$$

# Question 3C

**Problem Statement:** Suppose that a box contains four blue cards and four red cards. Suppose also that two of these eight cards are selected at random, without replacement. If it is known that at least one red card has been selected, what is the probability that both selected cards are red?

**Solution:**

1. **Define Events:**
   Let $R$ be the event that both selected cards are red.
   Let $A$ be the event that at least one red card is selected.

2. **Calculate** $Pr(R \cap A)$:
   The event $R \cap A$ is the same as the event $R$ because if both selected cards are red, then at least one red card has been selected.
   The probability of selecting both red cards from the 8 cards is:

$$Pr(R) = \frac{\binom{4}{2}\binom{4}{0}}{\binom{8}{2}} = \frac{6 \times 1}{28} = \frac{6}{28} = \frac{3}{14}$$

3. **Calculate** $Pr(A)$:
   Probability of selecting at least one red card:
   Calculate the probability of selecting no red cards (i.e., selecting two blue cards):

$$Pr(A^c) = \frac{\binom{4}{2}\binom{4}{0}}{\binom{8}{2}} = \frac{6}{28} = \frac{3}{14}$$

   Thus, the probability of selecting at least one red card is:

$$Pr(A) = 1 - Pr(A^c) = 1 - \frac{3}{14} = \frac{11}{14}$$

4. **Calculate** $Pr(R \mid A)$:
   By the definition of conditional probability:

$$Pr(R \mid A) = \frac{Pr(R \cap A)}{Pr(A)} = \frac{Pr(R)}{Pr(A)} = \frac{\frac{3}{14}}{\frac{11}{14}} = \frac{3}{11}$$

5. **Conclusion:**
   The probability that both selected cards are red, given that at least one red card has been selected, is:

$$\frac{3}{11} \approx 0.2727$$