STAT 131 Midterm 1 Solutions - Fall 2024

Instructor: Dr. Juhee Lee
TA: Antonio Aguirre

University of California, Santa Cruz
Fall 2024

Introduction

This document provides the solutions for Questions 1-6 of Quiz 1 for **STAT 131**, instructed by **Dr. Juhee Lee** at the University of California, Santa Cruz. Solutions were prepared by the TA, **Antonio Aguirre**.

Question 1 - A

Consider two events A and B with probabilities $\Pr(A) = 0.4$ and $\Pr(B) = 0.5$. We aim to find the probability that exactly one of the two events occurs under two different conditions.

General Setup

For any two events A and B, the probability that exactly one of them occurs can be calculated as:

$$\Pr(\text{exactly one of } A \text{ or } B) = \Pr(A) + \Pr(B) - 2 \cdot \Pr(A \cap B)$$

This expression can be derived by computing $\Pr(A \cup B) - \Pr(A \cap B)$.

Part (a): When A and B are Disjoint

Since A and B are disjoint events in this case, we know that:

$$\Pr(A \cap B) = 0$$

Substituting into our formula, we get:

$$\Pr(\text{exactly one of } A \text{ or } B) = \Pr(A) + \Pr(B) - 2 \cdot \Pr(A \cap B) = 0.4 + 0.5 = 0.9$$

Answer for Part (a): The probability that exactly one of the two events occurs when A and B are disjoint is 0.9.

Part (b): When $\Pr(A \cap B) = 0.15$

Here, A and B are not disjoint, and we are given that $\Pr(A \cap B) = 0.15$:

$$\Pr(\text{exactly one of } A \text{ or } B) = \Pr(A) + \Pr(B) - 2 \cdot \Pr(A \cap B)$$

Substitute the values:

$$= 0.4 + 0.5 - 2 \cdot 0.15 = 0.9 - 0.3 = 0.6$$

Answer for Part (b): The probability that exactly one of the two events occurs when $\Pr(A \cap B) = 0.15$ is 0.6.

Question 1 - B

Consider two events A and B with probabilities $\Pr(A) = 0.2$ and $\Pr(B) = 0.6$. We aim to find the probability that exactly one of the two events occurs under two different conditions.

General Setup

For any two events A and B, the probability that exactly one of them occurs can be calculated as:

$$\Pr(\text{exactly one of } A \text{ or } B) = \Pr(A) + \Pr(B) - 2 \cdot \Pr(A \cap B)$$

This expression can be derived by computing $\Pr(A \cup B) - \Pr(A \cap B)$.

Part (a): When A and B are Disjoint

When A and B are disjoint events, they cannot both happen simultaneously, so:

$$\Pr(A \cap B) = 0$$

Substituting this into our formula, we get:

$$\Pr(\text{exactly one of } A \text{ or } B) = \Pr(A) + \Pr(B) - 2 \cdot \Pr(A \cap B) = 0.2 + 0.6 - 2 \cdot 0 = 0.8$$

Answer for Part (a): The probability that exactly one of the two events occurs when A and B are disjoint is 0.8.

—

Part (b): When $\Pr(A \cap B) = 0.2$

In this case, A and B are not disjoint, and we are given that $\Pr(A \cap B) = 0.2$:

$$\Pr(\text{exactly one of } A \text{ or } B) = \Pr(A) + \Pr(B) - 2 \cdot \Pr(A \cap B)$$

Substitute the given values:

$$= 0.2 + 0.6 - 2 \cdot 0.2 = 0.8 - 0.4 = 0.4$$

Answer for Part (b): The probability that exactly one of the two events occurs when $\Pr(A \cap B) = 0.2$ is 0.4.

Question 1 - C

Consider two events A and B with probabilities $\Pr(A) = 0.35$ and $\Pr(B) = 0.4$. We aim to find the probability that exactly one of the two events occurs under two different conditions.

General Setup

For any two events A and B, the probability that exactly one of them occurs can be calculated as:

$$\Pr(\text{exactly one of } A \text{ or } B) = \Pr(A) + \Pr(B) - 2 \cdot \Pr(A \cap B)$$

This expression can be derived by computing $\Pr(A \cup B) - \Pr(A \cap B)$.

Part (a): When A and B are Disjoint

When A and B are disjoint events, they cannot both happen simultaneously, so:

$$\Pr(A \cap B) = 0$$

Substituting this into our formula, we get:

$$\Pr(\text{exactly one of } A \text{ or } B) = \Pr(A) + \Pr(B) - 2 \cdot \Pr(A \cap B) = 0.35 + 0.4 - 2 \cdot 0 = 0.75$$

Answer for Part (a): The probability that exactly one of the two events occurs when A and B are disjoint is 0.75.

—

Part (b): When $\Pr(A \cap B) = 0.10$

In this case, A and B are not disjoint, and we are given that $\Pr(A \cap B) = 0.10$:

$$\Pr(\text{exactly one of } A \text{ or } B) = \Pr(A) + \Pr(B) - 2 \cdot \Pr(A \cap B)$$

Substitute the given values:

$$= 0.35 + 0.4 - 2 \cdot 0.10 = 0.75 - 0.20 = 0.55$$

Answer for Part (b): The probability that exactly one of the two events occurs when $\Pr(A \cap B) = 0.10$ is 0.55.

Question 2 - A

A box contains 24 light bulbs, of which 4 are defective. A person selects 10 bulbs randomly from the box, and a second person takes the remaining 14. We need to find the probability that all four defective bulbs end up with either the first person or the second person.

Solution Outline

We define the following:

- D be the set of 4 defective bulbs.
- N be the set of 20 non-defective bulbs.

To find the desired probability, calculate the total number of ways to distribute the bulbs and identify the favorable cases.

Step 1: Total Ways to Select 10 Bulbs from 24

The total number of ways to select 10 bulbs from 24 is given by:

$$\binom{24}{10}$$

Step 2: Favorable Outcomes

There are two favorable cases:

- **Case 1**: The first person picks all four defective bulbs.
- **Case 2**: The second person picks all four defective bulbs.

Case 1: If the first person picks all four defective bulbs, they must also pick 6 additional non-defective bulbs. The number of ways to do this is:

$$\binom{4}{4}\binom{20}{6}\binom{14}{14} = \binom{20}{6}$$

Case 2: If the second person picks all four defective bulbs, then the first person picks only non-defective bulbs. The number of ways for the first person to pick 10 non-defective bulbs is:

$$\binom{20}{10}\binom{4}{4}\binom{10}{10} = \binom{20}{10}$$

The total number of favorable outcomes is the sum of these cases:

$$\text{Favorable Outcomes} = \binom{20}{6} + \binom{20}{10}$$

Step 3: Probability Calculation

The probability that all defective bulbs end up with either the first or second person is:

$$\frac{\binom{20}{6} + \binom{20}{10}}{\binom{24}{10}} \approx 0.0895$$

Question 2 - B

A box contains 20 light bulbs, of which 5 are defective. One person selects 8 bulbs randomly from the box, and a second person takes the remaining 12. We need to find the probability that all five defective bulbs end up with either the first person or the second person.

Solution Outline

We define the following:

- D: the set of 5 defective bulbs.
- N: the set of 15 non-defective bulbs.

To find the desired probability, calculate the total number of ways to distribute the bulbs and identify the favorable cases.

Step 1: Total Ways to Select 8 Bulbs from 20

The total number of ways to select 8 bulbs from 20 is given by:

$$\binom{20}{8}$$

Step 2: Favorable Outcomes

There are two favorable cases:

- **Case 1**: The first person picks all five defective bulbs.
- **Case 2**: The second person picks all five defective bulbs.

Case 1: If the first person picks all five defective bulbs, they must also pick 3 additional non-defective bulbs to complete their selection of 8 bulbs. The number of ways to do this is:

$$\binom{5}{5} \cdot \binom{15}{3} \cdot \binom{12}{12} = \binom{15}{3}$$

Case 2: If the second person picks all five defective bulbs, then the first person picks only non-defective bulbs. The number of ways for the first person to pick 8 non-defective bulbs is:

$$\binom{15}{8} \cdot \binom{5}{5} \cdot \binom{7}{7} = \binom{15}{8}$$

The total number of favorable outcomes is the sum of these cases:

$$\text{Favorable Outcomes} = \binom{15}{3} + \binom{15}{8}$$

Step 3: Probability Calculation

The probability that all defective bulbs end up with either the first or second person is:

$$\frac{\binom{15}{3} + \binom{15}{8}}{\binom{20}{8}}$$

Question 2 - C

A box contains 21 light bulbs, of which 3 are defective. One person selects 8 bulbs randomly from the box, and a second person takes the remaining 13. We need to find the probability that all three defective bulbs end up with either the first person or the second person.

Solution Outline

Define the following:

- D: the set of 3 defective bulbs.
- N: the set of 18 non-defective bulbs.

To find the desired probability, calculate the total number of ways to distribute the bulbs and identify the favorable cases.

Step 1: Total Ways to Select 8 Bulbs from 21

The total number of ways to select 8 bulbs from 21 is given by:

$$\binom{21}{8}$$

Step 2: Favorable Outcomes

There are two favorable cases:

- **Case 1**: The first person picks all three defective bulbs.
- **Case 2**: The second person picks all three defective bulbs.

Case 1: If the first person picks all three defective bulbs, they must also pick 5 additional non-defective bulbs to complete their selection of 8 bulbs. The number of ways to do this is:

$$\binom{3}{3} \cdot \binom{18}{5} \cdot \binom{13}{13} = \binom{18}{5}$$

Case 2: If the second person picks all three defective bulbs, then the first person picks only non-defective bulbs. The number of ways for the first person to pick 8 non-defective bulbs is:

$$\binom{18}{8} \cdot \binom{3}{3} \cdot \binom{10}{10} = \binom{18}{8}$$

The total number of favorable outcomes is the sum of these cases:

$$\text{Favorable Outcomes} = \binom{18}{5} + \binom{18}{8}$$

Step 3: Probability Calculation

The probability that all defective bulbs end up with either the first or second person is:

$$\frac{\binom{18}{5} + \binom{18}{8}}{\binom{21}{8}}$$

Question 3 - A

Suppose a box contains seven red balls and three blue balls, and four balls are randomly selected with replacements. Let X denote the number of red balls selected.

Part (a): Probability Function of X

The selection is with replacement, and each draw is independent. Since we are counting the number of red balls out of four draws, X follows a **Binomial distribution** with $n = 4$ and success probability $p = 0.7$.

The probability function of X is given by:

$$\Pr(X = k) = \binom{4}{k}(0.7)^k(0.3)^{4-k} \quad \text{for } k = 0, 1, 2, 3, 4$$

Part (b): Cumulative Distribution Function (c.d.f.) of X

The cumulative distribution function $F(x) = \Pr(X \leq x)$ for X, a **Binomial distribution** with $n = 4$ and success probability $p = 0.7$ is:

$$F(x) = \sum_{k=0}^{\min\{\lfloor x \rfloor, 4\}} \binom{4}{k}(0.7)^k(0.3)^{4-k}$$

Figure 1 and **Figure 2** show the plots for the pf and CDF respectively.

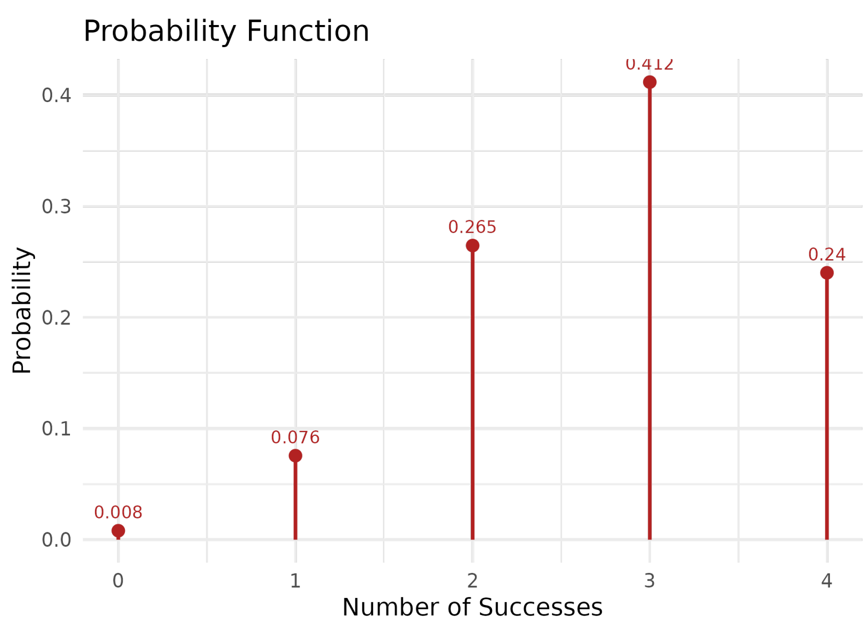

Figure 1: Probability Function (PF) of X

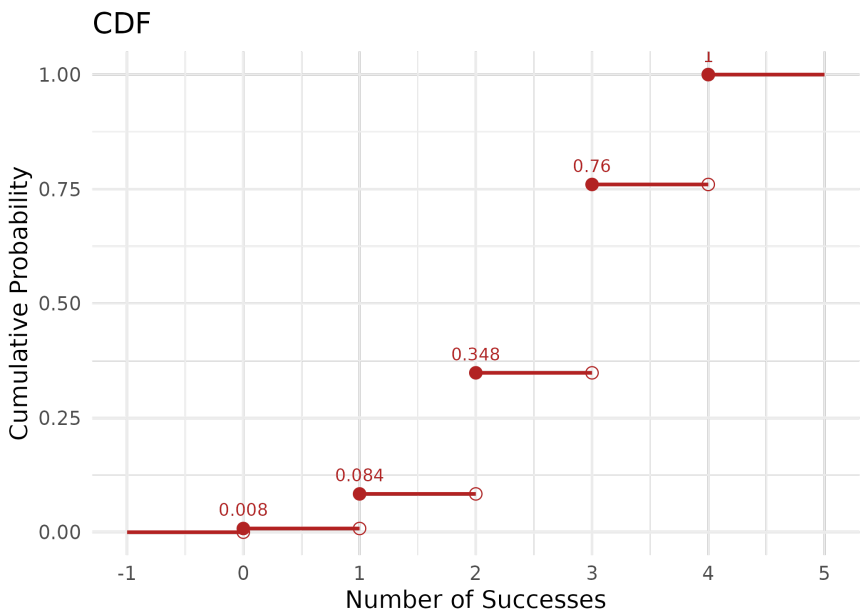

Figure 2: Cumulative Distribution Function (CDF) of X

Part (c): 0.5 Quantile (Median) of X

The 0.5 quantile, or median, is the smallest integer k for which $F(k) \geq 0.5$. Such an integer is achieved at $k = 3$, as it can be seen in **Figure 2**.

Question 3 - B

Suppose a box contains three red balls and seven blue balls, and four balls are selected at random, with replacement. Let X denote the number of red balls selected.

Part (a): Probability Function of X

Since the selection is with replacement and each draw is independent, the number of red balls X out of four draws follows a **Binomial distribution** with $n = 4$ and success probability $p = 0.3$.

The probability function of X is given by:

$$\Pr(X = k) = \binom{4}{k}(0.3)^k(0.7)^{4-k} \quad \text{for } k = 0, 1, 2, 3, 4$$

Part (b): Cumulative Distribution Function (c.d.f.) of X

The cumulative distribution function $F(x) = \Pr(X \leq x)$ for X, a **Binomial distribution** with $n = 4$ and success probability $p = 0.3$. is:

$$F(x) = \sum_{k=0}^{\min\{\lfloor x \rfloor, 4\}} \binom{4}{k}(0.3)^k(0.7)^{4-k}$$

Figure 1 and **Figure 2** show the plots for the pf and CDF respectively.

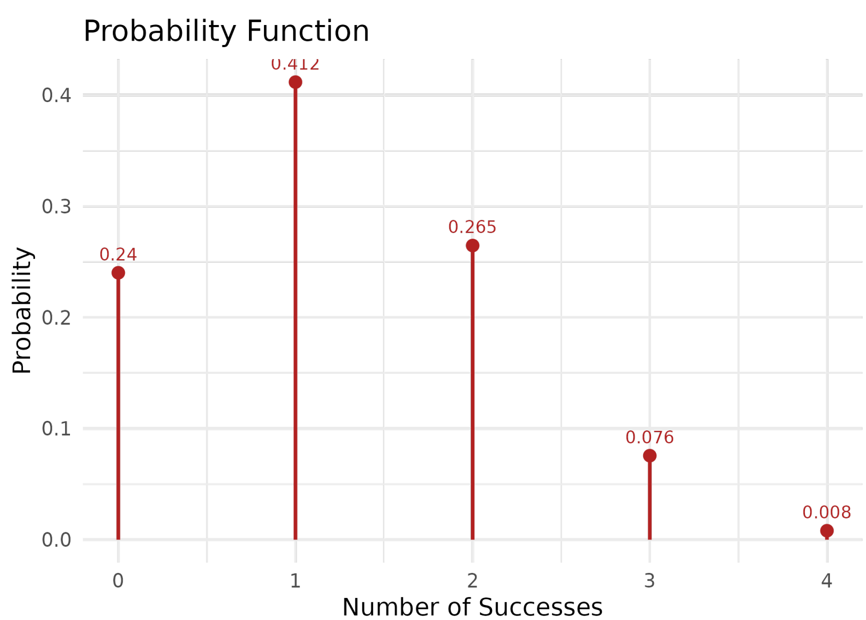

Figure 3: Probability Function (PF) of X

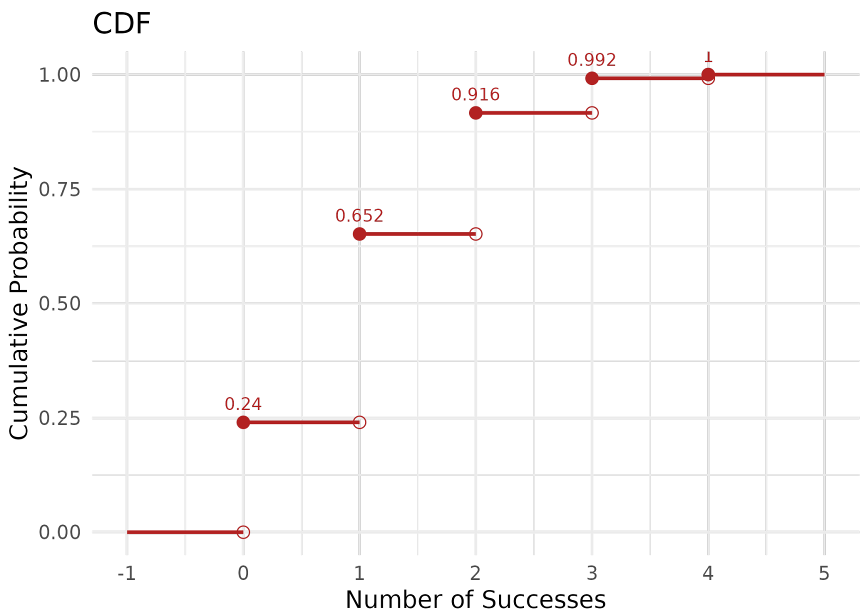

Figure 4: Cumulative Distribution Function (CDF) of X

Part (c): 0.4 Quantile (40th Percentile) of X

The 0.4 quantile is the smallest integer k such that $F(k) \geq 0.4$. We find this by calculating cumulative probabilities up to the value that first reaches or exceeds 0.4. For $X \sim \text{Binomial}(4, 0.3)$, this is achieved at $k = 1$.

Answer for Part (c): The 0.4 quantile of X is $k = 1$.

—

Question 3 - C

Suppose a box contains four red balls and six blue balls, and four balls are selected at random, with replacement. Let X denote the number of red balls selected.

Part (a): Probability Function of X

Since the selection is with replacement and each draw is independent, the number of red balls X out of four draws follows a **Binomial distribution** with $n = 4$ and success probability $p = 0.4$.

The probability function of X is given by:

$$\Pr(X = k) = \binom{4}{k}(0.4)^k(0.6)^{4-k} \quad \text{for } k = 0, 1, 2, 3, 4$$

Part (b): Cumulative Distribution Function (c.d.f.) of X

The cumulative distribution function $F(x) = \Pr(X \leq x)$ for X, a **Binomial distribution** with $n = 4$ and success probability $p = 0.4$, is:

$$F(x) = \sum_{k=0}^{\min\{\lfloor x \rfloor, 4\}} \binom{4}{k}(0.4)^k(0.6)^{4-k}$$

Figure 1 and **Figure 2** show the plots for the pf and CDF respectively.

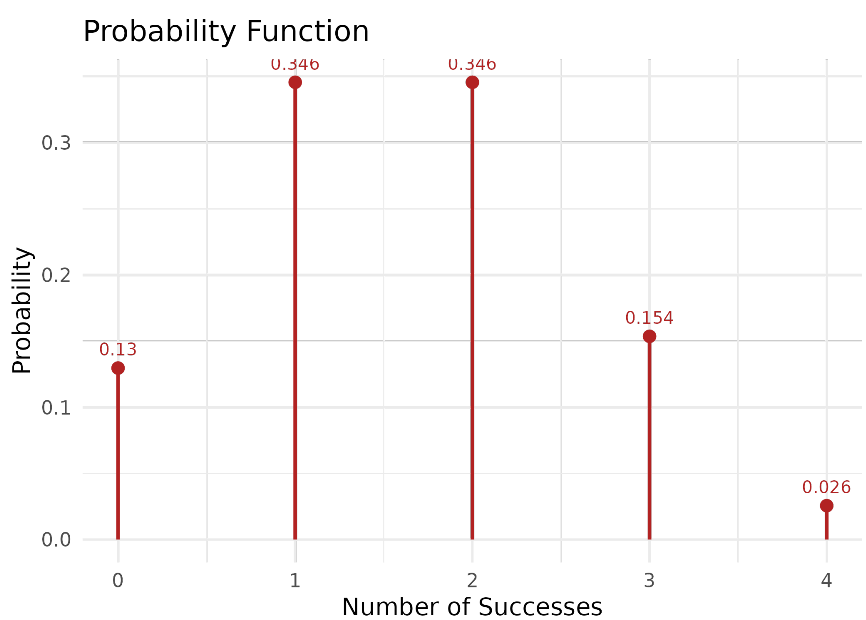

Figure 5: Probability Function (PF) of X

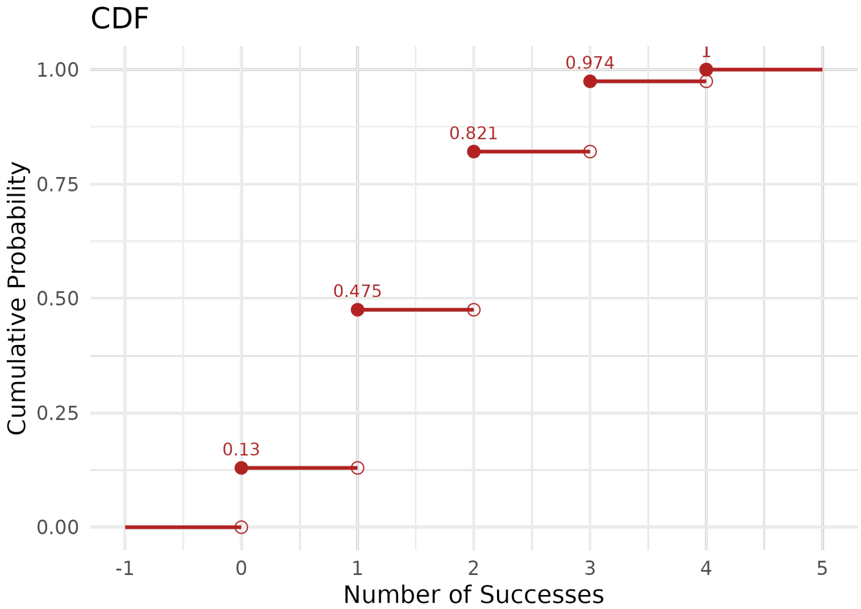

Figure 6: Cumulative Distribution Function (CDF) of X

Part (c): 0.6 Quantile (60th Percentile) of X

The 0.6 quantile is the smallest integer k such that $F(k) \geq 0.6$. We find this by calculating cumulative probabilities up to the value that first reaches or exceeds 0.6. For $X \sim \text{Binomial}(4, 0.4)$, this is achieved at $k = 2$.

Answer for Part (c): The 0.6 quantile of X is $k = 2$.

Question 4 - A

Dreamboat cars are produced at three factories: A, B, and C.

- **Factory A** produces 40% of the total Dreamboats,
- **Factory B** produces 30% of the total Dreamboats,
- **Factory C** produces 30% of the total Dreamboats.

Of the cars produced at each factory, the following percentages are lemons:

- 3% of cars from Factory A are lemons,
- 5% of cars from Factory B are lemons,
- 10% of cars from Factory C are lemons.

If you buy a Dreamboat and discover it is a lemon, we want to find the probability that it was produced at Factory B, i.e., $\Pr(B \mid L)$.

Step 1: Define Events and Given Probabilities

Define the following events:

- L: the event that a car is a lemon,
- A: the event that a car was produced at Factory A,
- B: the event that a car was produced at Factory B,
- C: the event that a car was produced at Factory C.

The given probabilities are:

1. **Production Probabilities:**

$$\Pr(A) = 0.4, \quad \Pr(B) = 0.3, \quad \Pr(C) = 0.3$$

2. **Conditional Probabilities of a Lemon:**

$$\Pr(L \mid A) = 0.03, \quad \Pr(L \mid B) = 0.05, \quad \Pr(L \mid C) = 0.10$$

Step 2: Calculate $\Pr(L)$ Using the Law of Total Probability

To find $\Pr(B \mid L)$, we first need $\Pr(L)$, the total probability of a randomly selected car being a lemon. Using the Law of Total Probability:

$$\Pr(L) = \Pr(L \mid A) \cdot \Pr(A) + \Pr(L \mid B) \cdot \Pr(B) + \Pr(L \mid C) \cdot \Pr(C)$$

Substitute the given values:

$$\Pr(L) = (0.03)(0.4) + (0.05)(0.3) + (0.10)(0.3)$$

Calculating each term separately:

$$\Pr(L) = 0.012 + 0.015 + 0.03 = 0.057$$

Thus, we have:

$$\Pr(L) = 0.057$$

Step 3: Apply Bayes' Theorem to Find $\Pr(B \mid L)$

To find the probability that a lemon came from Factory B, we apply Bayes' theorem:

$$\Pr(B \mid L) = \frac{\Pr(L \mid B) \cdot \Pr(B)}{\Pr(L)}$$

And therefore:

$$\Pr(B \mid L) = \frac{(0.05)(0.3)}{0.057} = \frac{0.015}{0.057} \approx 0.2632$$

Question 4 - B

Dreamboat cars are produced at three factories: A, B, and C.

- **Factory A** produces 40% of the total Dreamboats,
- **Factory B** produces 40% of the total Dreamboats,
- **Factory C** produces 20% of the total Dreamboats.

Of the cars produced at each factory, the following percentages are lemons:

- 3% of cars from Factory A are lemons,
- 2% of cars from Factory B are lemons,
- 10% of cars from Factory C are lemons.

If you buy a Dreamboat and discover it is a lemon, we want to find the probability that it was produced at Factory C, i.e., $\Pr(C \mid L)$.

Step 1: Define Events and Given Probabilities

Define the following events:

- L: the event that a car is a lemon,
- A: the event that a car was produced at Factory A,
- B: the event that a car was produced at Factory B,
- C: the event that a car was produced at Factory C.

The given probabilities are:

1. **Production Probabilities:**

$$\Pr(A) = 0.4, \quad \Pr(B) = 0.4, \quad \Pr(C) = 0.2$$

2. **Conditional Probabilities of a Lemon:**

$$\Pr(L \mid A) = 0.03, \quad \Pr(L \mid B) = 0.02, \quad \Pr(L \mid C) = 0.10$$

Step 2: Calculate $\Pr(L)$ Using the Law of Total Probability

To find $\Pr(C \mid L)$, we first need $\Pr(L)$, the total probability of a randomly selected car being a lemon. Using the Law of Total Probability:

$$\Pr(L) = \Pr(L \mid A) \cdot \Pr(A) + \Pr(L \mid B) \cdot \Pr(B) + \Pr(L \mid C) \cdot \Pr(C)$$

Substitute the given values:

$$\Pr(L) = (0.03)(0.4) + (0.02)(0.4) + (0.10)(0.2)$$

Calculating each term separately:

$$\Pr(L) = 0.012 + 0.008 + 0.02 = 0.04$$

Thus, we have:

$$\Pr(L) = 0.04$$

Step 3: Apply Bayes' Theorem to Find $\Pr(C \mid L)$

To find the probability that a lemon came from Factory C, we apply Bayes' theorem:

$$\Pr(C \mid L) = \frac{\Pr(L \mid C) \cdot \Pr(C)}{\Pr(L)}$$

Substitute the known values:

$$\Pr(C \mid L) = \frac{(0.10)(0.2)}{0.04} = \frac{0.02}{0.04} = 0.5$$

Answer: The probability that a lemon came from Factory C is 0.5 (or 50%).

—

Question 4 - C

Dreamboat cars are produced at three factories: A, B, and C.

- **Factory A** produces 30% of the total Dreamboats,
- **Factory B** produces 40% of the total Dreamboats,
- **Factory C** produces 30% of the total Dreamboats.

Of the cars produced at each factory, the following percentages are lemons:

- 3% of cars from Factory A are lemons,
- 2% of cars from Factory B are lemons,
- 10% of cars from Factory C are lemons.

If you buy a Dreamboat and discover it is a lemon, we want to find the probability that it was produced at Factory A, i.e., $\Pr(A \mid L)$.

Step 1: Define Events and Given Probabilities

Define the following events:

- L: the event that a car is a lemon,
- A: the event that a car was produced at Factory A,
- B: the event that a car was produced at Factory B,
- C: the event that a car was produced at Factory C.

The given probabilities are:

1. **Production Probabilities:**

$$\Pr(A) = 0.3, \quad \Pr(B) = 0.4, \quad \Pr(C) = 0.3$$

2. **Conditional Probabilities of a Lemon:**

$$\Pr(L \mid A) = 0.03, \quad \Pr(L \mid B) = 0.02, \quad \Pr(L \mid C) = 0.10$$

Step 2: Calculate $\Pr(L)$ Using the Law of Total Probability

To find $\Pr(A \mid L)$, we first need $\Pr(L)$, the total probability of a randomly selected car being a lemon. Using the Law of Total Probability:

$$\Pr(L) = \Pr(L \mid A) \cdot \Pr(A) + \Pr(L \mid B) \cdot \Pr(B) + \Pr(L \mid C) \cdot \Pr(C)$$

Substitute the given values:

$$\Pr(L) = (0.03)(0.3) + (0.02)(0.4) + (0.10)(0.3)$$

Calculating each term separately:

$$\Pr(L) = 0.009 + 0.008 + 0.03 = 0.047$$

Thus, we have:

$$\Pr(L) = 0.047$$

Step 3: Apply Bayes' Theorem to Find $\Pr(A \mid L)$

To find the probability that a lemon came from Factory A, we apply Bayes' theorem:

$$\Pr(A \mid L) = \frac{\Pr(L \mid A) \cdot \Pr(A)}{\Pr(L)}$$

Substitute the known values:

$$\Pr(A \mid L) = \frac{(0.03)(0.3)}{0.047} = \frac{0.009}{0.047} \approx 0.1915$$

Answer: The probability that a lemon came from Factory A is approximately 0.1915 (or 19.15%).

Question 5 - A

Suppose the p.d.f. of a random variable X is given by:

$$f(x) = \begin{cases} c(x+2) & \text{for } 0 \leq x \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Using an indicator function, we can rewrite the p.d.f. as:

$$f(x) = c(x+2)\mathbf{1}_{[0,3]}(x),$$

where $\mathbf{1}_{[0,3]}(x)$ is an indicator function that takes the value 1 if $x \in [0,3]$ and 0 otherwise.

Step 1: Set Up the Integral to Solve for c

Since $f(x)$ is a p.d.f., it must integrate to 1 over its entire range:

$$\int_0^3 c(x+2)\,dx = 1.$$

Step 2: Evaluate the Integral

Factor out c and integrate:

$$c\int_0^3 (x+2)\,dx = 1.$$

The integral of $x+2$ over $[0,3]$ is:

$$\int_0^3 (x+2)\,dx = \left[\frac{x^2}{2} + 2x\right]_0^3 = \frac{21}{2}.$$

Thus, we have:

$$c \cdot \frac{21}{2} = 1.$$

Step 3: Solve for c

Solve for c by dividing both sides by $\frac{21}{2}$:

$$c = \frac{2}{21}.$$

Question 5 - B

Suppose the p.d.f. of a random variable X is given by:

$$f(x) = \begin{cases} c(x+1) & \text{for } 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Using an indicator function, we can rewrite the p.d.f. as:

$$f(x) = c(x+1)\mathbf{1}_{[0,2]}(x),$$

where $\mathbf{1}_{[0,2]}(x)$ is an indicator function that takes the value 1 if $x \in [0, 2]$ and 0 otherwise.

Step 1: Set Up the Integral to Solve for c

Since $f(x)$ is a p.d.f., it must integrate to 1 over its range:

$$\int_0^2 c(x+1)\,dx = 1.$$

Step 2: Evaluate the Integral

Factor out c and integrate:

$$c\int_0^2 (x+1)\,dx = 1.$$

The integral of $x + 1$ over $[0, 2]$ is:

$$\int_0^2 (x+1)\,dx = \left[\frac{x^2}{2} + x\right]_0^2 = \frac{4}{2} + 2 = 4.$$

Thus, we have:

$$c \cdot 4 = 1.$$

Step 3: Solve for c

Solve for c by dividing both sides by 4:

$$c = \frac{1}{4}.$$

Answer: The value of the constant c is $\frac{1}{4}$.

—

Question 5 - C

Suppose the p.d.f. of a random variable X is given by:

$$f(x) = \begin{cases} c(x+2) & \text{for } 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Using an indicator function, we can rewrite the p.d.f. as:

$$f(x) = c(x+2)\mathbf{1}_{[0,2]}(x),$$

where $\mathbf{1}_{[0,2]}(x)$ is an indicator function that takes the value 1 if $x \in [0,2]$ and 0 otherwise.

Step 1: Set Up the Integral to Solve for c

Since $f(x)$ is a p.d.f., it must integrate to 1 over its range:

$$\int_0^2 c(x+2)\,dx = 1.$$

Step 2: Evaluate the Integral

Factor out c and integrate:

$$c\int_0^2 (x+2)\,dx = 1.$$

The integral of $x+2$ over $[0,2]$ is:

$$\int_0^2 (x+2)\,dx = \left[\frac{x^2}{2} + 2x\right]_0^2 = \frac{2}{2} + 4 = 5.$$

Thus, we have:

$$c \cdot 5 = 1.$$

Step 3: Solve for c

Solve for c by dividing both sides by 5:

$$c = \frac{1}{5}.$$

Answer: The value of the constant c is $\frac{1}{5}$.

Question 6 - A (Revised Solution)

The joint probability density function (p.d.f.) of X and Y is given by:

$$f(x,y) = \frac{3}{2}xy(2-x)\,\mathbf{1}_{(0,2)}(x)\,\mathbf{1}_{(0,1)}(y),$$

where $x \in (0,2)$ and $y \in (0,1)$.

Part (a): Find $\Pr(X < 1$ and $\frac{1}{2} < Y)$

To calculate $\Pr(X < 1$ and $\frac{1}{2} < Y)$, integrate the joint density over $x \in (0,1)$ and $y \in \left(\frac{1}{2}, 1\right)$:

$$\Pr(X < 1 \text{ and } \frac{1}{2} < Y) = \int_0^1 \int_{\frac{1}{2}}^1 f(x,y)\,dy\,dx.$$

Substitute $f(x,y) = \frac{3}{2}xy(2-x)$:

$$= \int_0^1 \int_{\frac{1}{2}}^1 \frac{3}{2}xy(2-x)\,dy\,dx.$$

Integrate with respect to y first:

$$= \int_0^1 \frac{3}{2}x(2-x) \int_{\frac{1}{2}}^1 y\,dy\,dx.$$

$$= \int_0^1 \frac{3}{2}x(2-x) \left[\frac{y^2}{2}\right]_{\frac{1}{2}}^1 dx.$$

Calculating the inner integral:

$$= \int_0^1 \frac{3}{2}x(2-x)\left(\frac{1}{2} - \frac{1}{8}\right) dx = \int_0^1 \frac{3}{2}x(2-x) \cdot \frac{3}{8}\,dx.$$

Simplify and continue integrating:

$$= \int_0^1 \frac{9}{16}x(2-x)\,dx = \frac{9}{16}\int_0^1 (2x - x^2)\,dx.$$

$$= \frac{9}{16}\left[x^2 - \frac{x^3}{3}\right]_0^1 = \frac{9}{16}\left(1 - \frac{1}{3}\right) = \frac{9}{16} \cdot \frac{2}{3} = \frac{3}{8}.$$

Answer for Part (a): $\Pr(X < 1$ and $\frac{1}{2} < Y) = \frac{3}{8}$.

Part (b): Find $\Pr(X < Y)$

To find $\Pr(X < Y)$, integrate over the region where $x < y$:

$$\Pr(X < Y) = \int_0^1 \int_x^1 f(x, y)\, dy\, dx.$$

Substitute $f(x, y) = \frac{3}{2}xy(2 - x)$:

$$= \int_0^1 \int_x^1 \frac{3}{2}xy(2 - x)\, dy\, dx.$$

Integrate with respect to y first:

$$= \int_0^1 \frac{3}{2}x(2 - x) \int_x^1 y\, dy\, dx.$$

$$= \int_0^1 \frac{3}{2}x(2 - x) \left[\frac{y^2}{2}\right]_x^1 dx.$$

Substitute the limits for y:

$$= \int_0^1 \frac{3}{2}x(2 - x) \left(\frac{1}{2} - \frac{x^2}{2}\right) dx.$$

Distribute and simplify:

$$= \int_0^1 \frac{3}{4}x(2 - x)(1 - x^2)\, dx.$$

Expanding terms:

$$= \int_0^1 \frac{3}{4}(2x - 3x^3 + x^5)\, dx.$$

Now integrate each term separately:

$$= \frac{3}{4}\left[x^2 - \frac{3x^4}{4} + \frac{x^6}{6}\right]_0^1.$$

Evaluating at $x = 1$:

$$= \frac{3}{4}\left(1 - \frac{3}{4} + \frac{1}{6}\right).$$

$$= \frac{3}{4} \cdot \frac{15}{24} = \frac{15}{32}.$$

Answer for Part (b): $\Pr(X < Y) = \frac{15}{32}$.

Part (c): Find $\Pr(X > 1)$

To find $\Pr(X > 1)$, integrate over $x \in (1, 2)$ and $y \in (0, 1)$:

$$\Pr(X > 1) = \int_1^2 \int_0^1 f(x, y)\, dy\, dx.$$

Substitute $f(x, y) = \frac{3}{2}xy(2 - x)$:

$$= \int_1^2 \int_0^1 \frac{3}{2}xy(2 - x)\, dy\, dx.$$

Integrate with respect to y:

$$= \int_1^2 \frac{3}{2}x(2 - x) \int_0^1 y\, dy\, dx.$$

$$= \int_1^2 \frac{3}{2}x(2 - x) \cdot \frac{1}{2}\, dx.$$

Simplify:

$$= \int_1^2 \frac{3}{4}x(2 - x)\, dx = \frac{3}{4} \int_1^2 (2x - x^2)\, dx.$$

Now integrate each term:

$$= \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_1^2.$$

Substitute the limits:

$$= \frac{3}{4}\left(4 - \frac{8}{3} - \left(1 - \frac{1}{3}\right)\right).$$

$$= \frac{3}{4}\left(3 - \frac{7}{3}\right) = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}.$$

Answer for Part (c): $\Pr(X > 1) = \frac{1}{2}$.

Question 6 - Version B

The joint probability density function (p.d.f.) of X and Y is given by:

$$f(x, y) = \frac{9}{4}xy(3-x)\,\mathbf{1}_{(0,3)}(x)\,\mathbf{1}_{(0,1)}(y),$$

where $x \in (0, 3)$ and $y \in (0, 1)$.

Part (a): Find $\Pr(X < 2$ and $\frac{1}{2} < Y)$

To calculate $\Pr(X < 2$ and $\frac{1}{2} < Y)$, we set up the integral over the region $x \in (0, 2)$ and $y \in \left(\frac{1}{2}, 1\right)$:

$$\Pr(X < 2, \frac{1}{2} < Y) = \int_0^2 \int_{\frac{1}{2}}^1 f(x, y)\, dy\, dx.$$

Substitute $f(x, y) = \frac{9}{4}xy(3-x)$:

$$= \int_0^2 \int_{\frac{1}{2}}^1 \frac{9}{4}xy(3-x)\, dy\, dx.$$

Integrate with respect to y:

$$= \int_0^2 \frac{9}{4}x(3-x) \int_{\frac{1}{2}}^1 y\, dy\, dx.$$

$$= \int_0^2 \frac{9}{4}x(3-x) \left[\frac{y^2}{2}\right]_{\frac{1}{2}}^1 dx.$$

Calculate the inner integral:

$$= \int_0^2 \frac{9}{4}x(3-x)\left(\frac{1}{2} - \frac{1}{8}\right) dx = \int_0^2 \frac{9}{4}x(3-x) \cdot \frac{3}{8}\, dx.$$

Simplify and continue:

$$= \int_0^2 \frac{27}{32}x(3-x)\, dx = \frac{27}{32}\int_0^2 (3x - x^2)\, dx.$$

$$= \frac{27}{32}\left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^2 = \frac{27}{32}\left(3 - \frac{8}{3}\right) = \frac{27}{32} \cdot \frac{1}{3} = \frac{9}{32}.$$

Answer for Part (a): $\Pr(X < 2$ and $\frac{1}{2} < Y) = \frac{9}{32}$.

—

Part (b): Find $\Pr(X < Y)$

To find $\Pr(X < Y)$, we integrate over the region where $x < y$:

$$\Pr(X < Y) = \int_0^1 \int_x^1 f(x, y)\, dy\, dx.$$

Substitute $f(x, y) = \frac{9}{4}xy(3 - x)$:

$$= \int_0^1 \int_x^1 \frac{9}{4}xy(3 - x)\, dy\, dx.$$

Integrate with respect to y:

$$= \int_0^1 \frac{9}{4}x(3 - x) \int_x^1 y\, dy\, dx.$$

$$= \int_0^1 \frac{9}{4}x(3 - x) \left[\frac{y^2}{2}\right]_x^1 dx.$$

Evaluate the inner integral:

$$= \int_0^1 \frac{9}{4}x(3 - x) \left(\frac{1}{2} - \frac{x^2}{2}\right) dx.$$

$$= \int_0^1 \frac{9}{8}x(3 - x)(1 - x^2)\, dx.$$

Expanding terms:

$$= \int_0^1 \frac{9}{8}(3x - 3x^3 - x^2 + x^4)\, dx.$$

Now integrate each term:

$$= \frac{9}{8}\left[\frac{3x^2}{2} - \frac{3x^4}{4} - \frac{x^3}{3} + \frac{x^5}{5}\right]_0^1.$$

$$= \frac{9}{8}\left(\frac{3}{2} - \frac{3}{4} - \frac{1}{3} + \frac{1}{5}\right).$$

Calculating each term:

$$= \frac{9}{8} \cdot \frac{30}{40} = \frac{111}{160}.$$

Answer for Part (b): $\Pr(X < Y) = \frac{111}{160}$.

—

Part (c): Find $\Pr(X > 2)$

To find $\Pr(X > 2)$, integrate $f(x, y)$ over $x \in (2, 3)$ and $y \in (0, 1)$:

$$\Pr(X > 2) = \int_2^3 \int_0^1 f(x, y)\, dy\, dx.$$

Substitute $f(x, y) = \frac{9}{4}xy(3 - x)$:

$$= \int_2^3 \int_0^1 \frac{9}{4}xy(3 - x)\, dy\, dx.$$

Integrate with respect to y:

$$= \int_2^3 \frac{9}{4}x(3 - x) \int_0^1 y\, dy\, dx.$$

$$= \int_2^3 \frac{9}{4}x(3 - x) \cdot \frac{1}{2}\, dx.$$

Simplify:

$$= \int_2^3 \frac{9}{8}x(3 - x)\, dx = \frac{9}{8} \int_2^3 (3x - x^2)\, dx.$$

Now integrate each term:

$$= \frac{9}{8} \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_2^3.$$

Evaluate at the bounds:

$$= \frac{9}{8} \left(\frac{27}{2} - \frac{27}{3} - \left(\frac{12}{2} - \frac{8}{3} \right) \right).$$

$$= \frac{9}{8} \left(\frac{27 - 18}{2} \right) = \frac{1}{2}.$$

Answer for Part (c): $\Pr(X > 2) = \frac{1}{2}$.

Question 6 - Version C

The joint probability density function (p.d.f.) of X and Y is given by:

$$f(x,y) = \frac{3}{2}xy(2-y)\,\mathbf{1}_{(0,1)}(x)\,\mathbf{1}_{(0,2)}(y),$$

where $x \in (0,1)$ and $y \in (0,2)$.

Part (a): Find $\Pr\left(X < \frac{1}{2} \text{ and } 1 < Y\right)$

To find $\Pr\left(X < \frac{1}{2} \text{ and } 1 < Y\right)$, we integrate over the regions $x \in \left(0, \frac{1}{2}\right)$ and $y \in (1,2)$:

$$\Pr\left(X < \frac{1}{2} \text{ and } 1 < Y\right) = \int_0^{\frac{1}{2}} \int_1^2 f(x,y)\,dy\,dx.$$

Substitute $f(x,y) = \frac{3}{2}xy(2-y)$:

$$= \int_0^{\frac{1}{2}} \int_1^2 \frac{3}{2}xy(2-y)\,dy\,dx.$$

Integrate with respect to y:

$$= \int_0^{\frac{1}{2}} \frac{3}{2}x \int_1^2 y(2-y)\,dy\,dx.$$

Expanding and integrating:

$$= \int_0^{\frac{1}{2}} \frac{3}{2}x \left[y^2 - \frac{y^3}{3}\right]_1^2 dx.$$

Evaluate the inner integral:

$$= \int_0^{\frac{1}{2}} \frac{3}{2}x \left(4 - \frac{8}{3} - \left(1 - \frac{1}{3}\right)\right) dx = \int_0^{\frac{1}{2}} \frac{3}{2}x \cdot \frac{1}{2}\,dx.$$

$$= \int_0^{\frac{1}{2}} \frac{3}{4}x\,dx = \frac{3}{4}\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} = \frac{3}{4} \cdot \frac{1}{8} = \frac{3}{32}.$$

Answer for Part (a): $\Pr\left(X < \frac{1}{2} \text{ and } 1 < Y\right) = \frac{3}{32}$.
—

Part (b): Find $\Pr(X > Y)$

We calculate $\Pr(X > Y)$ over the region where $x > y$:

$$\Pr(X > Y) = \int_0^2 \int_0^y f(x,y)\,dx\,dy.$$

Substitute $f(x,y) = \frac{3}{2}xy(2-y)$:

$$= \int_0^2 \int_0^y \frac{3}{2}xy(2-y)\,dx\,dy.$$

Integrate with respect to x:

$$= \int_0^2 \frac{3}{2}y(2-y) \int_0^y x\,dx\,dy.$$

Evaluating the inner integral:

$$= \int_0^2 \frac{3}{2}y(2-y) \left[\frac{x^2}{2}\right]_0^y dy.$$

$$= \int_0^2 \frac{3}{2}y(2-y) \cdot \frac{y^2}{2}\,dy = \int_0^2 \frac{3}{4}y^3(2-y)\,dy.$$

Expanding and integrating:

$$= \int_0^2 \frac{3}{4}(2y^3 - y^4)\,dy = \frac{3}{4}\left[\frac{2y^4}{4} - \frac{y^5}{5}\right]_0^1.$$

Evaluating at the bounds:

$$= \frac{3}{4}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{4} \cdot \frac{3}{10} = \frac{9}{40}.$$

Answer for Part (b): $\Pr(X > Y) = \frac{9}{40}$.
—

Part (c): Find $\Pr(Y > 1)$

To find $\Pr(Y > 1)$, we integrate $f(x,y)$ over $y \in (1,2)$ and $x \in (0,1)$:

$$\Pr(Y > 1) = \int_0^1 \int_1^2 f(x,y)\,dy\,dx.$$

Substitute $f(x,y) = \frac{3}{2}xy(2-y)$:

$$= \int_0^1 \int_1^2 \frac{3}{2}xy(2-y)\,dy\,dx.$$

Integrate with respect to y:

$$= \int_0^1 \frac{3}{2}x \int_1^2 y(2-y)\,dy\,dx.$$

Expanding and integrating:

$$= \int_0^1 \frac{3}{2}x \left[y^2 - \frac{y^3}{3}\right]_1^2 dx.$$

Evaluating at the bounds:

$$= \int_0^1 \frac{3}{2}x \cdot \frac{1}{2}\,dx = \frac{3}{4}\int_0^1 x\,dx = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}.$$

Answer for Part (c): $\Pr(Y > 1) = \frac{3}{8}$.