Conditional Distributions Review

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Winter 2025

Introduction

This document provides a review of conditional distributions and marginalization techniques in probability theory. The goal is to clarify the concepts of conditional densities, variable roles, and evaluation techniques.

1. Overview of Notation

- **Function of a Variable**: $f(x)$ means that x is the variable.
- **Function Evaluated at a Constant**: $f(x = a)$ or simply $f(a)$ fixes x at a, resulting in a constant.

Example of Function vs. Evaluation

- **Function**: $f(x) = x^2$
- **Evaluation**: $f(3) = 3^2 = 9$

2. Conditional Density for Random Variables

Given random variables X, Y, and Z with a joint density $f_{X,Y,Z}(x, y, z)$:

Conditional Density Notation

$$f_{A|B=b}(a) = \frac{f_{A,B}(a, b)}{f_B(b)}$$

3. Examples

Example 1: Single Variable A, Two Variables B

Goal: Compute $f_{X|Y=y,Z=z}(x)$.

1. **Joint Density:** $f_{X,Y,Z}(x, y, z)$.
2. **Marginal Density:**
$$f_{Y,Z}(y, z) = \int_{-\infty}^{\infty} f_{X,Y,Z}(u, y, z)\, du$$
3. **Conditional Density Formula:**
$$f_{X|Y=y,Z=z}(x) = \frac{f_{X,Y,Z}(x, y, z)}{f_{Y,Z}(y, z)}$$

4. General Tips for Marginalizing

- Marginalizing out variables:
$$f_X(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y,Z}(x, y, z)\, dy\, dz$$

- For joint marginal densities:

$$f_{X,Y}(x,y) = \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dz$$

5. General Formula Recap

Key Formula

$$f_{A|B=b}(a) = \frac{f_{A,B}(a,b)}{f_B(b)}$$

- Identify the **variable** (A) and the **conditioning event** $(B = b)$.
- Compute the **marginal density** $f_B(b)$ by integrating over all other variables.

Exercise

Given: $f_{X,Y,Z}(x, y, z) = c(x + 2y + 3z)$, where $x, y, z \in [0, 1]$.

Questions

1. Find c such that $f_{X,Y,Z}(x, y, z)$ is a valid density.
2. Compute the marginal densities:
$$f_X(x), \quad f_Y(y), \quad f_Z(z)$$
3. Compute the joint marginals:
$$f_{X,Y}(x, y), \quad f_{X,Z}(x, z), \quad f_{Y,Z}(y, z)$$
4. Compute the conditional density $f_{X|Y=y,Z=z}(x)$ and evaluate:
$$f_{X|Y=0.1,Z=0.2}(x), \quad f_{X|Y=0.5,Z=0.5}(x)$$
5. Compute:
$$\Pr(X > 0.1|Y = 0.5, Z = 0.7)$$