Sample space: Ω

Event: $A \subseteq \Omega$

Random variable: $X : \Omega \to \mathbb{R}$

Support(X) := set of plausible values of X

Supp(X)

neither

Countable

$\{0,1\}$
$\{1,2,3,4,5,6\}$
$\{0,1,2,\dots\}$

not countable

$[0,1]$
$(0,\infty)$
$(-\infty,\infty)$

X is discrete

pmf

$f(x) := \Pr(X=x)$

$\hookrightarrow \sum_{x \in Supp(X)} f(x) = 1$

X is continious

pdf

- Given to you, $f(x)$

$\hookrightarrow \int_{x \in Supp(X)} f(x)\,dx = 1$

- $f(x) \neq \Pr(X=x) = 0$

Example: rolling 2 dice

$\Omega := \{$

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$\}$

$A := \{$outcomes $\in \Omega$ | sum of outcomes is 3$\} = \{(1,2),(2,1)\}$

$X :=$ sum of outcomes

- $\{X=12\} = \{(6,6)\}$
- $\{X=3\} = A$
- $\{X=x\}$ are events!

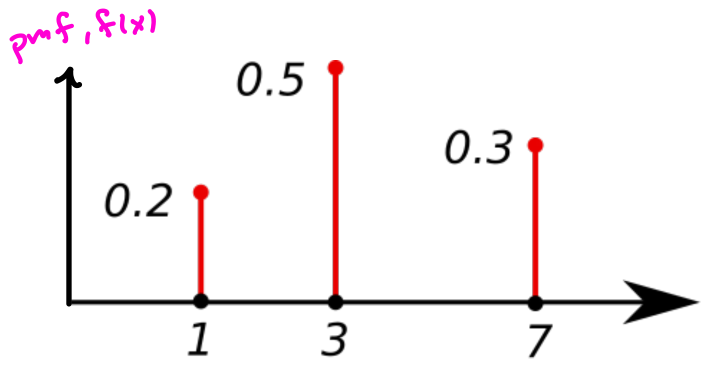
Example: earthquakes in CA

$\Omega := \{$all earthquakes in CA$\}$

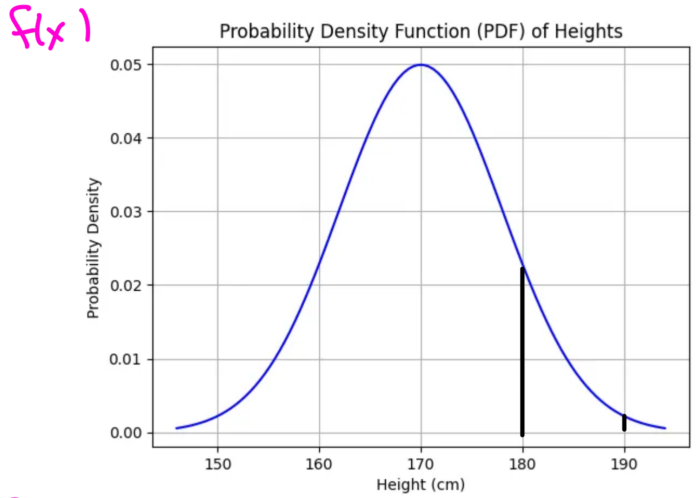
$A := \{$earthquakes $\in \Omega$ | earthquake with a mag. larger than 6$\}$

$X :=$ magnitude of earthquakes

- $\{X=3\}$
- $\{X \in [4, 6.5]\}$
- $\{X \in (6,\infty]\} = A$



$f(3) = Pr(X=3) = 0.5$ | $f(2) = 0$
$f(1) + f(3) + f(7) = 1$



$f(160) = 0.023$ $Pr(X \in (180, 190))$

Table 1: Overview of Selected Probability Distributions. By Antonio Aguirre.

Distribution	Expression	Support	Typical Phenomenon Modeled
Discrete			
Binomial	$P(X = k) = nkp^k(1-p)^{n-k}$	$k = 0, 1, 2, \dots, n$	Number of successes in n trials (e.g., coin flips)
Poisson	$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$	$k = 0, 1, 2, \dots$	Number of events in a fixed interval (e.g., calls per hour)
Geometric	$P(X = k) = (1-p)^{k-1}p$	$k = 1, 2, 3, \dots$	Number of trials until first success (e.g., failures before success)
Continuous			
Normal	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$x \in R$	Measurement errors, IQ scores, anything with a natural symmetric variation
Exponential	$f(x) = \lambda e^{-\lambda x}$	$x \geq 0$	Time between events in a Poisson process (e.g., time between bus arrivals)
Uniform	$f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$	$x \in [a, b]$	Equal likelihood of continuous outcomes (e.g., random decimals between 0 and 1)

2 Random Variables and Distribution Functions

2.1 Discrete and continuous distributions

[redacted] [redacted]

3. Suppose that a random variable X has a discrete distribution with the following p.f.:

$f(3) = \frac{c}{2^3}$

$$f(x) = \begin{cases} \frac{c}{2^x}, & \text{for } x = 0, 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Supp(X)

Find the value of the constant c.

Def

$$1 = \sum_{x=0}^{\infty} f(x)$$
$$= \sum_{x=0}^{\infty} \frac{c}{2^x}$$
$$= c\sum_{x=0}^{\infty} \frac{1}{2^x}$$
$$= c\sum_{x=0}^{\infty} \left(\frac{1}{2}\right)^x$$

Geom Series

$$= c\,\frac{1}{1-1/2}$$
$$= c\,2$$

$r \in (0,1)$

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}.$$

$Pr(X=3) = f(3) = \frac{c}{2^3} = \frac{1}{2}\cdot\frac{1}{2^3} = \frac{1}{16}$

$Pr(X=\pi) = 0$

Then $c = 1/2$

continious
Random variables

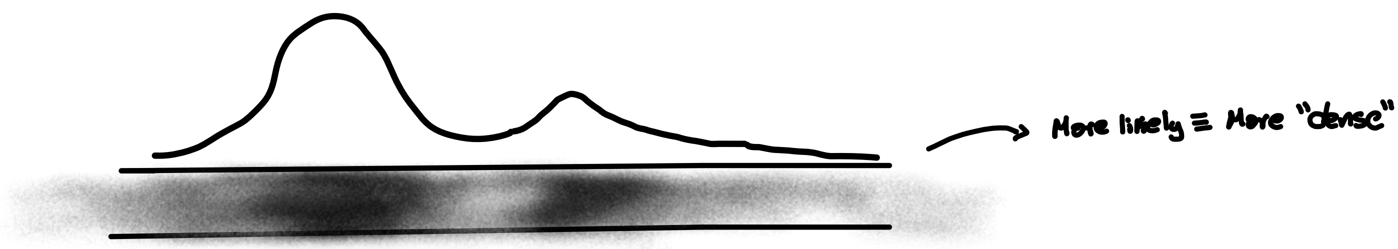
Definition 3.11 (continuous random variable)**.** The random variable X is continuous, if $\exists$ a function $f : \mathbb{R} \to \mathbb{R}_+$ such that

$$\boxed{P(a \leq X \leq b) = \int_a^b f(x)dx.}$$

How is pdf related to events' probabilities? For continuous random variable X, the probability of event E is

$$\mathbb{P}(E) = \int_{x \in E} f(x)dx.$$

A probability density function is not probability. For continuous random variable X, $P(X = a) = 0$ but its pdf at a can be strictly positive.



Example 3.9 (Weather)**.** A nice day is defined as the temperature is between 60 F and 68 F. Given the pdf of tomorrow's temperature, find the probability that tomorrow is a nice day. ♢

Properties of pdf.

- Non-negative: $f(x) \geq 0$ for all $x \in \mathbb{R}$.
- Unity: $\int_{-\infty}^{\infty} f(x)dx = 1$.

check at 3:48 : https://youtu.be/hDjcxi9p0ak?si=HGkZtVNjkwZFMLzN

4. Suppose that the p.d.f. of a random variable X is as follows:

Support

$$f(x) = \begin{cases} cx^2, & \text{for } 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$f(x) = cx^2 \, \mathbb{1}_{(1,2)}(x)$$

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

(a) Find the value of the constant c and sketch the p.d.f.

(b) Find the value of $Pr(X > 3/2)$.

a) [redacted]

b) Pr [redacted]

$$= 3/7 \int_{3/2}^{\infty} x^2 \, \mathbb{1}_{(1,2)}(x) \, dx \qquad Pr\left(X \in (3/2, \infty) \right)$$

$$= 3/7 \int_{3/2}^{2} x^2 \, dx \qquad Pr\left(X \in (\tfrac{3}{2}, 2) \right)$$

5. Suppose that the p.d.f. of a random variable X is as follows:

$$f(x) = \begin{cases} \frac{1}{8}x, & \text{for } 0 \le x \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of t such that $Pr(X \le t) = 1/4$.

(b) Find the value of t such that $Pr(X \ge t) = 1/2$.

$\boxed{\text{For you!}}$

Hint: · Find an expression for $Pr(X \ge t)$ in terms of t

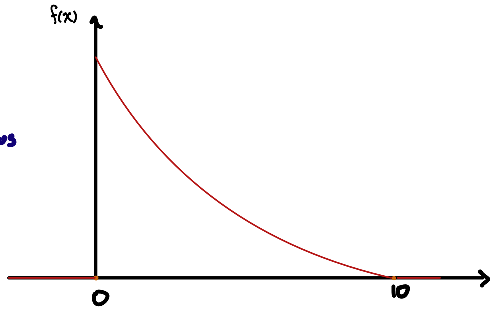
· Notice $Pr(X \le t) = 1 - Pr(X \ge t)$

2.2 CDF :

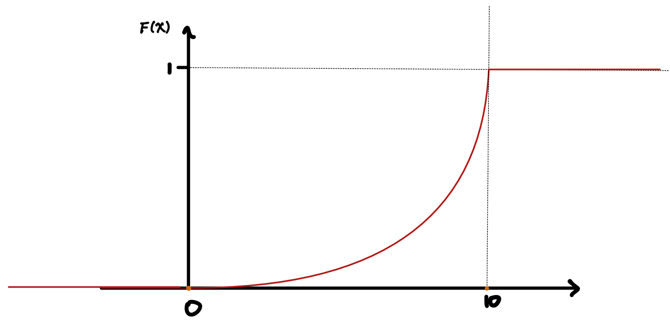
$\boxed{\text{CDF of } X: \quad F(x) = Pr(X \leq x)}$

$$CDF(x) = \int_{-\infty}^{x} f(x)\,dx$$

X is continuous

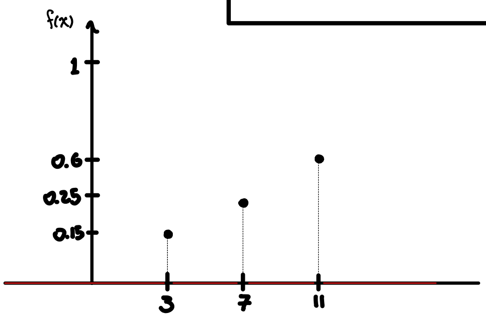


$\longleftrightarrow$

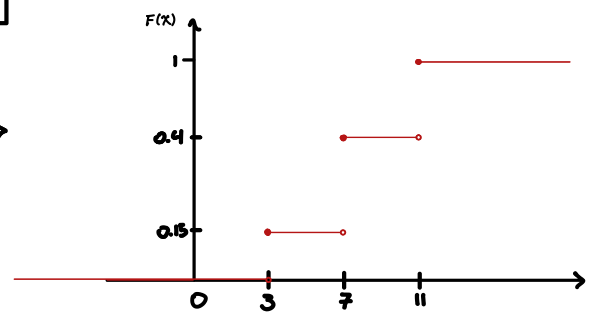


$\boxed{\text{CDF} \quad F(x) = Pr(X \leq x) = \sum^{x} f(x)}$

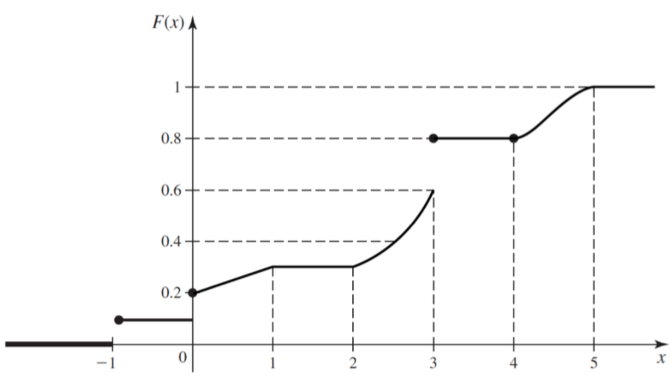
X is discrete



$\longleftrightarrow$



1. Suppose that the CDF F of a random variable X is as sketched in the following figure.



(a) $Pr(X = -1)$

(b) $Pr(X < 0)$

(c) $Pr(X \leq 0)$

(d) $Pr(X = 1)$

(e) $Pr(0 < X \leq 3)$

(f) $Pr(0 < X < 3)$

(g) $Pr(0 \leq X \leq 3)$

(h) $Pr(1 < X \leq 2)$

(i) $Pr(1 \leq X \leq 2)$

(j) $Pr(X > 5) = 1 - Pr(X \leq 5)$

(k) $Pr(X \geq 5)$

(l) $Pr(3 \leq X \leq 4)$

$\boxed{\text{For you!}}$

$$Pr(a < X \leq b) = F(b) - F(a)$$

2. Suppose that the c.d.f. of a random variable X is as follows:

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{1}{9}x^2, & \text{for } 0 \leq x \leq 3, \\ 1, & \text{for } x > 3. \end{cases}$$

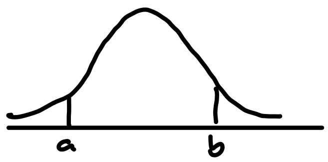
(a) Find and sketch the p.d.f. of X.

(b) Find the quantile function.

$\boxed{\text{For you!}}$

Hint for b) : I. Find the CDF of f
II. Find the inverse of the CDF

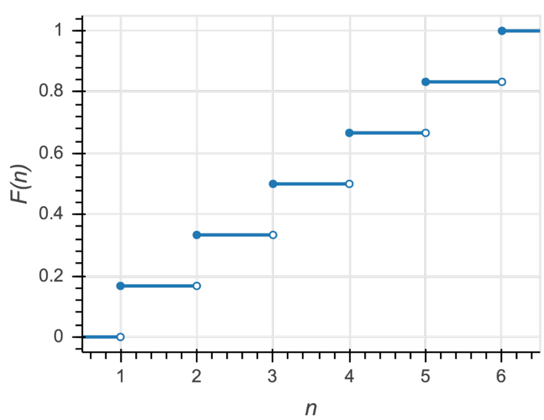
* Also, check the extra exercises I attached.



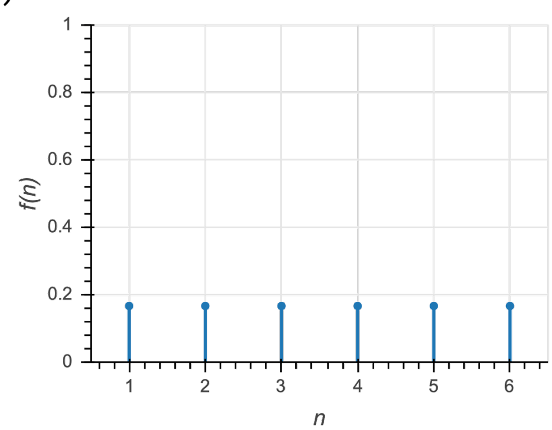
$$\Pr(a < X \leq b) = \int_a^b f(x)dx = \int_{-\infty}^b f(x)dx - \int_{-\infty}^a f(x)dx$$
$$= F(b) - F(a)$$

$$\Pr(X=-1) = \Pr(-2 < X \leq -1)$$
$$= F(-1) - F(-2)$$

a)



b)



More?

↓

3. Find the quantile function for the given CDF as follows:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 0 & \text{for } x \le 0, \\ \int_0^x \frac{dy}{(1+y)^2} & \text{for } x > 0, \end{cases} = \int_0^x \frac{1}{(1+y)^2} dy$$

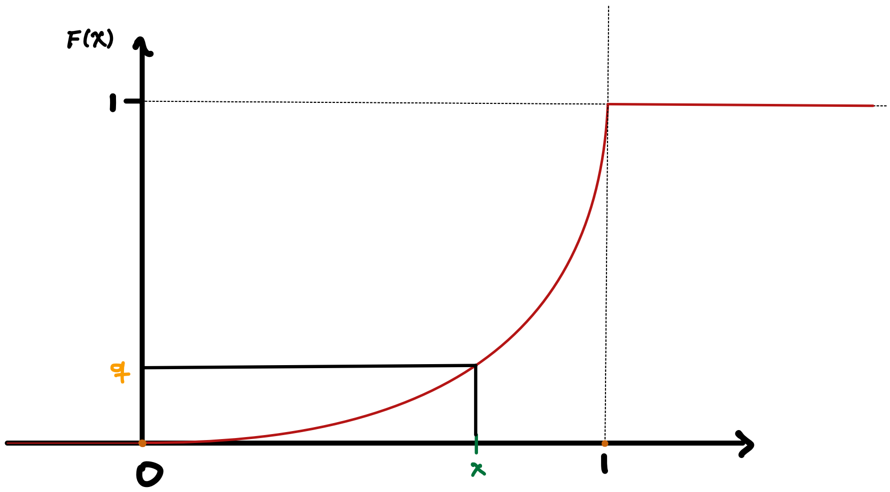
What is the 90'th quantile?

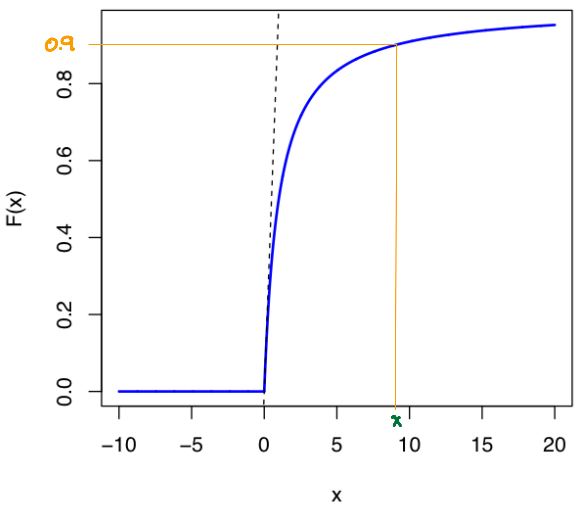
Recap

The q-th quantile of $F(x)$ is:

$$\boxed{x \text{ s.t. } q = F(x)}$$

Note: The quantile function is just the inverse of the CDF.





I. $0.9 = \int_0^{x} \frac{1}{(1+y)^2}\,dy$

$= -(1+y)^{-1}\Big|_0^{x}$

$= (1+y)^{-1}\Big|_x^{0}$

$= 1-(1+x)^{-1}$

$0.9 = 1-(1+x)^{-1}$

$x = \frac{1}{(1-0.9)} - 1$

$\boxed{\text{whats the quantile function?}}$