Poisson Process: Web Server Traffic Example

Antonio Aguirre
University of California, Santa Cruz

Introduction

The **Poisson process** is a fundamental model used to describe the arrival of events occurring randomly over time. A classic application is modeling web server traffic, where requests from users arrive unpredictably.

Two equivalent ways to describe a Poisson process are:

1. **Counting Process Perspective:** Tracks the total number of arrivals over a given time period.
2. **Interarrival Time Perspective:** Focuses on the random time intervals between consecutive arrivals.

Both perspectives lead to useful insights. We explore this through an example of web server requests later in this document.

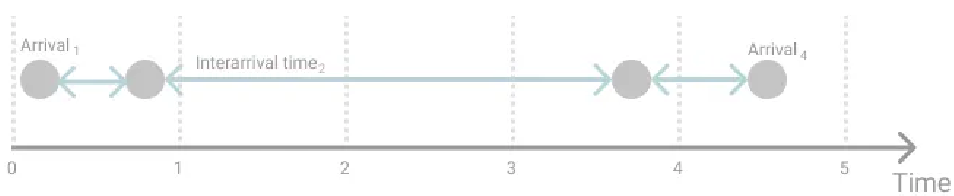

Figure 1: Illustration of a Poisson Process. Notice in here there are four arrivals up to time five, and three waiting times between arrivals.

Formal Definition of a Poisson Process

A stochastic process $\{N(t)\}_{t\geq 0}$ is a **Poisson process** with rate $\lambda > 0$ if it satisfies:

1. **Initial Condition:** $N(0) = 0$, meaning no requests have arrived at $t = 0$.
2. **Independent Increments:** The number of arrivals in non-overlapping time intervals are independent.

3. **Poisson-Distributed Increments:** The number of arrivals in a time interval $(s, t]$ follows a Poisson distribution:

$$P(N(t) - N(s) = k) = \frac{(\lambda(t-s))^k e^{-\lambda(t-s)}}{k!}, \quad k = 0, 1, 2, \dots$$

Characterization via Exponential Interarrival Times

An alternative but equivalent way to define a Poisson process is through the **interarrival times**—the time intervals between consecutive arrivals. Instead of focusing on how many events occur in a given time interval, we shift our perspective to analyzing the waiting time until the next arrival.

Definition of Interarrival Times

Let $T_1, T_2, \dots$ represent the time intervals between successive arrivals in the Poisson process. That is, if the first event occurs at time S_1, the second at S_2, and so on, then the interarrival times are given by:

$$T_1 = S_1, \quad T_2 = S_2 - S_1, \quad T_3 = S_3 - S_2, \quad \dots$$

where S_n represents the arrival time of the nth event.

A fundamental result states that these interarrival times are independent and follow an **exponential distribution** with rate λ, denoted as:

$$T_i \sim \text{Exp}(\lambda).$$

This means the probability that the time until the next arrival exceeds t is:

$$P(T_i > t) = e^{-\lambda t}, \quad t \geq 0.$$

Memoryless Property and Its Consequences

The exponential distribution exhibits the important **memoryless property**, which states that:

$$P(T > t + s \mid T > s) = P(T > t).$$

This implies that if no request has arrived by time s, the remaining waiting time until the next arrival is still exponentially distributed with the same parameter λ. In other words, the process "forgets" how long it has already waited, making it a natural model for scenarios where arrivals occur randomly and independently over time.

Equivalence Between the Two Definitions

The Poisson process can be equivalently characterized in two ways:

- **Poisson Counting Perspective:** The number of arrivals in time t follows a Poisson distribution:

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \ldots$$

- **Exponential Interarrival Time Perspective:** The waiting times between consecutive arrivals are i.i.d. exponential random variables:

$$T_i \sim \text{Exp}(\lambda).$$

Equivalence

- If the number of arrivals in time t follows a Poisson distribution, then the time between consecutive arrivals must be exponentially distributed.
- Conversely, if interarrival times are i.i.d. exponential, then the total number of arrivals up to time t follows a Poisson distribution.

Why is This Useful in Practice?

The dual characterization of the Poisson process is powerful because it allows flexibility in modeling:

- The **counting perspective** is useful for analyzing how many events occur in a fixed period, e.g., estimating web server loads over time.
- The **interarrival perspective** helps in understanding the timing between events, e.g., optimizing server response times.

Example: Web Server Requests

Suppose a web server receives requests at an average rate of $\lambda = 12$ requests per second.

1. Counting Process Perspective

We can determine the probability of receiving exactly 10 requests in a given second:

$$P(N(1) = 10) = \frac{(12)^{10} e^{-12}}{10!} \approx 0.104.$$

This tells us how likely it is that exactly 10 users access the server within a second.

2. Interarrival Time Perspective

We can also determine the probability that the next request arrives after 0.2 seconds:

$$P(T_1 > 0.2) = e^{-12 \times 0.2} = e^{-2.4} \approx 0.091.$$

This shows that a long wait between requests is rare due to the high arrival rate.

Preventing Server Overload

Suppose our web server has a maximum capacity of 1000 requests per minute. This means we are interested in calculating:

$$P(N(1 \text{ min}) \geq 1000)$$

Alternatively, we could use the interarrival time formulation. Since a total of 1000 requests means summing 1000 independent interarrival times, we check:

$$P\left(\sum_{i=1}^{1000} T_i \geq 1 \text{ min}\right).$$

If the sum exceeds 1 minute, the server remains stable; otherwise, it overloads.

Notice in both cases, I'm trying to answer the same question: **What is the likelihood of my server being overload?**

Other Common Applications of Poisson Processes

The Poisson process is widely used in various domains to model random events occurring over time. Some notable applications include:

- **Hospital Emergency Room Arrivals:** Patient arrivals in an emergency room can be modeled as a Poisson process, allowing for effective resource allocation based on varying arrival rates throughout the day.
- **Call Center Requests:** Incoming customer calls in a call center often follow a Poisson process, helping managers optimize staffing schedules to handle peak periods efficiently.
- **Radioactive Decay:** The emission of particles from a radioactive substance occurs randomly over time and is well-described by a Poisson process (used for radiation detection).
- **Astronomy and Astrophysics:** The detection of photons or cosmic rays arriving at a telescope follows a Poisson process used for signal processing in deep-space observations.
- **Traffic Flow Analysis:** Vehicles arriving at a toll booth or crossing an intersection at random intervals can be studied using Poisson models to optimize urban transportation planning.