STAT 131 — Discussion (Lec 12–14)

Prepared for students

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Quick facts you'll use

- **Variance algebra:**
  $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2, \quad \mathrm{Var}(X \pm Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) \pm 2\,\mathrm{Cov}(X,Y).$
- **Covariance of independent r.v.'s:**
  If $X \perp Y$, then $\mathrm{Cov}(X,Y) = 0$.
- **Order statistics for exponentials:**
  For n i.i.d. $\mathrm{Exp}(\lambda)$, the gap times between consecutive failures are independent exponentials with rates $n\lambda, (n-1)\lambda, \dots$.
- **Moment generating function (mgf):**
  $M_X(t) = \mathbb{E}[e^{tX}]$. If X, Y independent, $M_{aX+bY+c}(t) = e^{ct} M_X(at)\, M_Y(bt)$.
- **Uniform mgf and moments:**
  If $X \sim \mathrm{Unif}(a,b)$,

$$M_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}\ (t \neq 0), \quad M_X(0) = 1, \quad \mathbb{E}[X] = \tfrac{a+b}{2},\ \mathrm{Var}(X) = \tfrac{(b-a)^2}{12}.$$

Discussion [Lec 12]

1. Show $\mathbb{E}[(X-Y)^2] = \mathrm{Var}(X) + \mathrm{Var}(Y)$ when $X \perp Y$ and $\mathbb{E}[X] = \mathbb{E}[Y]$

Setup. Expand the square and take expectations:

$$\mathbb{E}[(X-Y)^2] = \mathbb{E}[X^2] - 2\,\mathbb{E}[XY] + \mathbb{E}[Y^2].$$

Independence gives $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$. Let $\mu := \mathbb{E}[X] = \mathbb{E}[Y]$. Then

$$\mathbb{E}[(X-Y)^2] = \big(\mathrm{Var}(X) + \mu^2\big) - 2\mu^2 + \big(\mathrm{Var}(Y) + \mu^2\big) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

Remark. Independence is stronger than needed; it suffices that $\mathrm{Cov}(X,Y) = 0$.

2. n independent items with lifetimes $\mathrm{Exp}(\lambda)$: expected time until 3 failures

Key fact. With n i.i.d. exponentials, the waiting time to the first failure is $\mathrm{Exp}(n\lambda)$, then to the second is $\mathrm{Exp}((n-1)\lambda)$, then to the third is $\mathrm{Exp}((n-2)\lambda)$.

$$\mathbb{E}[T_{3\text{ failures}}] = \frac{1}{n\lambda} + \frac{1}{(n-1)\lambda} + \frac{1}{(n-2)\lambda}.$$

$$\boxed{\mathbb{E}[T] = \frac{1}{\lambda}\Big(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2}\Big)}.$$

Discussion [Lec 13]

1. Random word length from "the girl put on her beautiful red hat"

Words: the(3), girl(4), put(3), on(2), her(3), beautiful(9), red(3), hat(3). Thus $X \in \{2,3,4,9\}$ with counts $(1,5,1,1)$ and equal word probability $1/8$.

$$\mathbb{E}[X] = \frac{2+5\cdot 3+4+9}{8} = \frac{30}{8} = 3.75 = \boxed{\tfrac{15}{4}}.$$

Given $X \le 4$, we restrict to lengths $\{2,3,4\}$ with counts $(1,5,1)$:

$$\mathbb{E}[X \mid X \le 4] = \frac{2+5\cdot 3+4}{7} = \frac{21}{7} = \boxed{3}.$$

2. $\theta \sim \mathrm{Unif}(0,2\pi)$; $X = \cos\theta$, $Y = \sin\theta$

$$\mathbb{E}[X] = \mathbb{E}[\cos\theta] = 0, \quad \mathbb{E}[Y] = \mathbb{E}[\sin\theta] = 0,$$
$$\mathbb{E}[XY] = \mathbb{E}[\tfrac{1}{2}\sin(2\theta)] = 0 \;\Rightarrow\; \mathrm{Cov}(X,Y) = 0.$$

But $X^2 + Y^2 = 1$ a.s., a deterministic constraint, so X and Y are *not* independent.

$$\boxed{\mathrm{Cov}(X,Y) = 0 \text{ but } X \not\perp Y.}$$

Discussion [Lec 14]

1. MGF of $\mathrm{Unif}(a,b)$; use it to find mean and s.d.

MGF. For $X \sim \mathrm{Unif}(a,b)$,

$$M_X(t) = \mathbb{E}[e^{tX}] = \frac{1}{b-a}\int_a^b e^{tx}\,\mathrm{d}x = \frac{e^{tb}-e^{ta}}{t(b-a)} \quad (t \neq 0), \qquad M_X(0) = 1.$$

Mean via mgf. $M_X'(t) = \frac{\mathrm{d}}{\mathrm{d}t} M_X(t)$, so $\mathbb{E}[X] = M_X'(0)$. Differentiating and taking $t \to 0$ (e.g. by l'Hôpital) yields $\mathbb{E}[X] = \dfrac{a+b}{2}$.

$$\boxed{\mathbb{E}[X] = \frac{a+b}{2}.}$$

Variance via mgf. $\mathbb{E}[X^2] = M_X''(0)$; compute (or recall) $\mathrm{Var}(X) = \dfrac{(b-a)^2}{12}$. Hence $\mathrm{sd}(X) = \dfrac{b-a}{\sqrt{12}} = \dfrac{b-a}{2\sqrt{3}}$.

$$\boxed{\mathrm{Var}(X) = \frac{(b-a)^2}{12}, \quad \mathrm{sd}(X) = \frac{b-a}{\sqrt{12}}.}$$

2. If X, Y iid with mgf $\psi(t) = e^{t^2+3t}$, find mgf of $Z = 2X - 3Y + 4$

By independence and linearity:

$$M_Z(t) = e^{4t}\, M_X(2t)\, M_Y(-3t) = e^{4t}\, \psi(2t)\, \psi(-3t) = e^{4t}\, e^{(2t)^2+3(2t)}\, e^{(-3t)^2+3(-3t)} = e^{13t^2+t}.$$

$$\boxed{M_Z(t) = \exp\bigl(13t^2 + t\bigr).}$$

Interpretation. Match to the Normal mgf $e^{\mu t + \frac{1}{2}\sigma^2 t^2}$: $\mu = 1$, $\sigma^2 = 26$. So $Z \sim \mathcal{N}(1, 26)$.