STAT 131 — Discussion 11 Hints & Solutions

Prepared for students
University of California, Santa Cruz

Quick reminders

Discrete vs. continuous: the wisest quick test is to check the *support*. Countable set (e.g. $\{-3,\ldots,3\}$) $\Rightarrow$ pmf. Intervals/areas (e.g. $0 < x < y < 9$) $\Rightarrow$ pdf.

From a joint pdf to marginals: if $f_{X,Y}(x,y)$ is given on a region R, then

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x$$

Inverse CDF method: If $U \sim \mathrm{Unif}(0,1)$ and G is a target CDF, then $Y = G^{-1}(U)$ has that distribution.

1. Call center: marginals from a triangular support

Given: joint pdf

$$f_{X,Y}(x,y) = e^{-y} \quad \text{on } 0 < x < y < \infty, \text{ and 0 elsewhere.}$$

(Here X is time on hold; Y is total call time.)

How to read the pictures

- The shaded wedge is the support $R = \{(x,y) : 0 < x < y\}$ (we show a truncated window for visibility).
- **Left:** fix a y (horizontal line). Inside R the allowed x runs from 0 to y, so $f_Y(y) = \int_0^y e^{-y}\,\mathrm{d}x = ye^{-y}$ for $y > 0$.
- **Right:** fix an x (vertical line). Inside R the allowed y runs from x to ∞, so $f_X(x) = \int_x^\infty e^{-y}\,\mathrm{d}y = e^{-x}$ for $x > 0$.

$$\boxed{f_Y(y) = y\,e^{-y}\ (y > 0) \qquad \text{and} \qquad f_X(x) = e^{-x}\ (x > 0).}$$

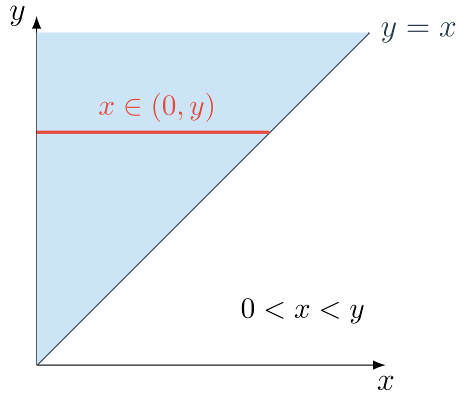

Figure 1: *
Horizontal slice for $f_Y(y) = \int_0^y f_{X,Y}(x,y)\,\mathrm{d}x$.

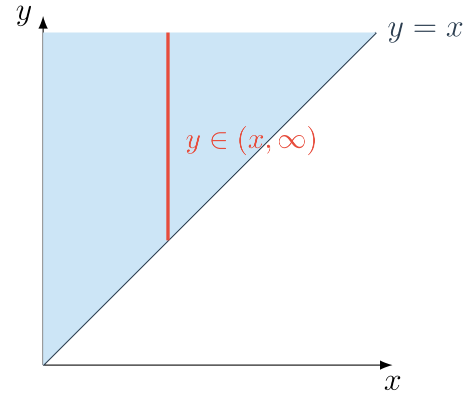

Figure 2: *
Vertical slice for $f_X(x) = \int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y$.

2. Discrete transform: $X \in \{-3,-2,-1,0,1,2,3\}$ equally likely, $Y = X^2 - X$

Each x has probability $1/7$. Map $x \mapsto y = x^2 - x$:

x	-3	-2	-1	0	1	2	3
$y = x^2 - x$	12	6	2	0	0	2	6

Collect equal outputs:

$P(Y=0) = \frac{2}{7}\ (x=0,1),\ P(Y=2) = \frac{2}{7}\ (x=-1,2),\ P(Y=6) = \frac{2}{7}\ (x=-2,3),\ P(Y=12) = \frac{1}{7}\ (x=-3)$

Hence the pmf is

$$\boxed{p_Y(y) = \begin{cases} 2/7, & y \in \{0,2,6\}, \\ 1/7, & y = 12, \\ 0, & \text{otherwise.} \end{cases}}$$

3. Build Y with pdf $g(y) = \frac{3}{8}y^2$ on $(0, 2)$ from $X \sim$ Unif$(0, 1)$

We use the inverse CDF method.

Step 1 (target CDF).

$$G(y) = \int_0^y \frac{3}{8}t^2 \, \mathrm{d}t = \frac{y^3}{8}, \qquad 0 < y < 2.$$

Step 2 (invert). Set $U \in (0, 1)$ and solve $U = G(y) = y^3/8$:

$$y = (8U)^{1/3}.$$

Step 3 (define the transform). If $X \sim \text{Unif}(0, 1)$ and we set

$$\boxed{Y = r(X) = (8X)^{1/3},}$$

then by the inverse CDF theorem Y has pdf $g(y) = \frac{3}{8}y^2$ on $(0, 2)$.

Check (optional)

Differentiate $G(y) = y^3/8$ to recover $g(y)$; or apply the 1D change–of–variables formula with $r^{-1}(y) = y^3/8$.